ANATOMY

THE HARMONIC BOUNDARY VALUE PROBLEM FOR AN ELLIPSE OR AN ELLIPSOID

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1. Introduction. The solution of Laplace's equation in the unit circle taking on given boundary values on the circumference can be represented by a series of the elementary harmonic functions $r^n \cos n\phi$, $r^n \sin n\phi$, in terms of the polar coordinates (r, ϕ), with coefficients determined by expanding the boundary values in a Fourier series.* The series is not without importance, even though the solution can be alternatively expressed in "closed" form by means of Poisson's integral. This paper† is concerned with the explicit details of a modification of the series by which it can be adapted to the solution of the corresponding problem for an arbitrary ellipse. The method is simple in principle and is such as to lend itself readily to actual calculation.

A similar transition is indicated from the solution of the boundary value problem for a sphere by means of Laplace series to a solution for an arbitrary ellipsoid; the formulas, necessarily much more complicated, are however not carried through to the same degree of explicitness.

The reader who is familiar with the theory of orthogonal curvilinear coordinates and Lamé series will recognize that there is room for further discussion as to the relation between that theory and the elementary approach followed here.

2. Solution of the boundary value problem for an ellipse in series of special harmonic functions. Let the ellipse be given by the parametric representation

$$x = a \cos \theta, \qquad y = b \sin \theta, \qquad 0 \leqq \theta \leqq 2\pi, \qquad a > b. \tag{1}$$

This θ is of course not the polar angle in a polar coordinate system; the latter will enter into the discussion at a later stage. But it is easy to calculate the value of θ corresponding to an assigned point of the ellipse, or the coordinates corresponding to any assigned θ.

* See for example D. Jackson, Fourier Series and Orthogonal Polynomials, Carus Mathematical Monographs, No. 6, 1941, pp. 103–105.

† The paper is an "expository" one, in the sense that no single feature of it can lay claim to any appreciable degree of novelty. The facts were however in the present instance worked out by the writer during a period of restriction to the house by illness, and he does not know where to lay his hand on a connected account of them in the literature. The results of §4 can be obtained more compactly by use of the terminology of "confocal coordinates"; the formulation adopted here, on the other hand, avoids certain questions of detail associated with the introduction of those coordinates in general terms. The relation between §5 and the theory of Lamé series appears to be matter for further study.

For a comprehensive theory of developments in series of harmonic polynomials with which the problem treated here is associated in its general aspects see J. L. Walsh, The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions, Bulletin of the American Mathematical Society, vol. 35, 1929, pp. 499–544; G. M. Merriman, On the expansion of harmonic functions in terms of normal-orthogonal harmonic polynomials, American Journal of Mathematics, vol. 53, 1931, pp. 589–596.

Let the boundary values to which a harmonic function is to be fitted be specified by a function $f(\theta)$. This function can be expanded in a Fourier series

$$\frac{a_0}{2} + a_1 \cos\theta + a_2 \cos 2\theta + \cdots + b_1 \sin\theta + b_2 \sin 2\theta + \cdots \tag{2}$$

which will converge uniformly to $f(\theta)$ if $f(\theta)$ is continuous and satisfies appropriate further hypotheses.

Each of the functions $\cos n\theta$, $\sin n\theta$ can be expressed as a polynomial of the nth degree in $\cos\theta$ and $\sin\theta$, hence in x/a and y/b, or, with different coefficients, as a polynomial of the nth degree in x and y, where x and y are associated with θ by the equations (1). That is to say, there is for each of the functions $\cos n\theta$, $\sin n\theta$ a polynomial of the nth degree in x and y which reduces on the ellipse to the cosine or sine of the multiple of θ in question. There are in fact infinitely many such polynomials in each case, for $n \geqq 2$, since values on the ellipse are unaffected by addition of any polynomial containing $(x/a)^2+(y/b)^2-1$ as a factor.

There are just $2n+1$ polynomials of the nth or lower degree which are linearly independent on the ellipse. For any polynomial can be reduced by use of the relation $y^2=b^2[1-(x/a)^2]$ to a form in which no term contains a higher power of y than the first; that is, any polynomial of the nth or lower degree can be expressed on the ellipse as a linear combination of the $2n+1$ monomials

$$1,\ x,\ y,\ x^2,\ xy,\ x^3,\ x^2y,\ \cdots,\ x^n,\ x^{n-1}y.$$

On the other hand, the constant 1 and the $2n$ harmonic polynomials forming the real and imaginary components of $(x+iy)^k$, $k=1, 2, \cdots, n$, are linearly independent on the ellipse, or on any closed curve in the plane. For if a linear combination of them with coefficients not all zero were to vanish on the curve it would be a harmonic function vanishing on the curve but not vanishing identically in its interior, in contradiction with the theorem of uniqueness for the boundary value problem. Consequently any polynomial of the nth degree can be expressed *on the ellipse* as a linear combination of these $2n+1$ harmonic polynomials. Let

$$u_0 = 1, \quad u_1 = x, \quad v_1 = y, \quad u_2 = x^2 - y^2, \quad v_2 = 2xy,$$

and in general let $(x+iy)^k=u_k(x, y)+iv_k(x, y)$, each of the functions u_k, v_k being a polynomial of the kth degree in the two variables together. In particular the polynomials representing $\cos n\theta$ and $\sin n\theta$ can be linearly expressed on the ellipse in terms of the u's and v's; there are uniquely determined linear combinations of $u_0, u_1, v_1, \cdots, u_n, v_n$ which reduce on the ellipse to $\cos n\theta$ and $\sin n\theta$ respectively. Let these combinations, which are themselves harmonic polynomials in x and y, be denoted by $U_n(x, y)$ and $V_n(x, y)$. It is to be shown presently that the calculation of their coefficients can readily be carried through in detail.

The existence of a unique solution of Laplace's equation taking on the boundary values $f(\theta)$ being assumed as known, let $F(x, y)$ denote this solution. Let $s_n(\theta)$ denote the partial sum of the series (2) through terms of the nth order, and let

$$S_n(x, y) = \frac{a_0}{2} + a_1U_1(x, y) + a_2U_2(x, y) + \cdots + a_nU_n(x, y) + b_1V_1(x, y) + b_2V_2(x, y) + \cdots + b_nV_n(x, y).$$

Then $S_n(x, y)$ is a harmonic polynomial reducing on the ellipse to $s_n(\theta)$. If $f(\theta)$ is such that $s_n(\theta)$ converges uniformly toward $f(\theta)$, and if $\epsilon > 0$, then

$$|f(\theta) - s_n(\theta)| < \epsilon$$

for $0 \leqq \theta \leqq 2\pi$, when n is sufficiently large. As the difference $f(\theta) - s_n(\theta)$ represents the boundary values of the harmonic function $F(x, y) - S_n(x, y)$ on the ellipse, it follows from the maximum-minimum theorem that

$$|F(x, y) - S_n(x, y)| < \epsilon$$

throughout the interior of the ellipse, for the same values of n. When the U's and V's are known, the solution $F(x, y)$ of the boundary value problem has the series representation

$$F(x, y) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_kU_k(x, y) + b_kV_k(x, y)],$$

the a's and b's being merely the ordinary Fourier coefficients of $f(\theta)$; and the convergence of the series throughout the interior of the ellipse is at least as rapid as that of the Fourier series for the boundary values.

By the least-square property of the Fourier series, $S_n(x, y)$ as an approximation to $F(x, y)$ is the harmonic polynomial of the nth degree at most such that the integral of the square of the error around the boundary is a minimum *for integration with respect to* θ. An approximation satisfying a different least-square criterion, for example that of minimizing the integral with respect to arc length (*cf.* Merriman, *loc. cit.*), could be obtained in theory by the use of orthogonal trigonometric sums with an appropriate weight function* in place of the simple monomials $\cos n\theta$ and $\sin n\theta$, but would not be so readily accessible to calculation.

3. Calculation of the special functions by the method of undetermined coefficients. The sequence of U's and V's begins with

$$U_0 = 1 = u_0, \qquad U_1 = x/a = u_1/a, \qquad V_1 = y/b = v_1/b,$$

which are equal to 1, $\cos\theta$, $\sin\theta$ on the ellipse. Also $V_2 = 2xy/(ab) = v_2/(ab)$ is the harmonic polynomial corresponding to $\sin 2\theta = 2 \sin\theta \cos\theta$. If the polynomial

* See D. Jackson, Orthogonal trigonometric sums, Annals of Mathematics, (2), vol. 34, 1933, pp. 799–814.

$(x^2/a^2)-(y^2/b^2)$, representing $\cos 2\theta=\cos^2\theta-\sin^2\theta$ on the curve, is expressed with undetermined coefficients as a linear combination of $u_2=x^2-y^2$ and $(x^2/a^2)+(y^2/b^2)$, it is found on evaluation of the coefficients that

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{1}{a^2+b^2}\left[2(x^2-y^2)+(b^2-a^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\right],$$

which means for points on the ellipse, since $(x^2/a^2)+(y^2/b^2)=1$, that

$$(a^2+b^2)\cos 2\theta = 2u_2-(a^2-b^2),$$

that is to say, the polynomial U_2 is defined by the identity

$$(a^2+b^2)U_2 = 2u_2-(a^2-b^2)u_0.$$

By similar but progressively less simple calculations, which the reader need not repeat unless he cares to do so, inasmuch as an alternative method of verification is presently to be indicated, it is found that

$$\begin{aligned}(a^3+3ab^2)U_3 &= 4u_3-3(a^2-b^2)u_1,\\ (3a^2b+b^3)V_3 &= 4v_3-3(a^2-b^2)v_1,\\ (a^4+6a^2b^2+b^4)U_4 &= 8u_4-8(a^2-b^2)u_2+(a^2-b^2)^2u_0,\\ (4a^3b+4ab^3)V_4 &= 8v_4-8(a^2-b^2)v_2.\end{aligned} \tag{3}$$

In these identities a general law of formation begins to be recognizable empirically, especially if a symbol $v_0\equiv 0$ is introduced for the sake of uniformity, so that a term $(a^2-b^2)^2v_0$ can be added to the right-hand member of the last relation, while the one giving V_2 can be written in the form

$$2abV_2 = 2v_2-(a^2-b^2)v_0.$$

The parentheses by which U_n and V_n are multiplied on the left contain alternate terms of the binomial expansion of $(a+b)^n$, while the coefficients on the right, apart from the powers of a^2-b^2 in regular succession, are the coefficients of the polynomial expressing $\cos n\theta$ in terms of $\cos\theta$.

It is to be shown that this law of formation is indeed general, not by a formal proof in combinatorial terms, but by indication of a method of approach from which it will be apparent how such a proof could be constructed.

4. Calculation of the special functions by the algebra of complex quantities. Let the polar coordinates of an arbitrary point of the plane be (r, ϕ), connected with x and y by the equations

$$x = r\cos\phi, \qquad y = r\sin\phi,$$

so that on the ellipse

$$a\cos\theta = r\cos\phi, \qquad b\sin\theta = r\sin\phi. \tag{4}$$

The problem is to express $\cos n\theta$ and $\sin n\theta$ in terms of $r^k \cos k\phi$ and $r^k \sin k\phi$, $k=0, 1, \cdots, n$, by means of the last pair of relations; since

$$(x + iy)^k = r^k(\cos k\phi + i \sin k\phi),$$

the functions $r^k \cos k\phi$ and $r^k \sin k\phi$ are identical with the u_k and v_k of the earlier paragraphs. The resulting expressions will be linear combinations of the u's and v's which reduce to $\cos n\theta$ and $\sin n\theta$ when the point (r, ϕ) is on the ellipse. The equation of the ellipse no longer enters except as it is implicit in the parametric representation. It is to be borne in mind that a and b are constant, while θ, r, and ϕ are variable. The question as thus presented is merely one of algebraic and trigonometric manipulation based on the relations (4).

Let (real) quantities m and γ be defined so that

$$a = m \cosh \gamma, \qquad b = m \sinh \gamma, \tag{5}$$

that is, let

$$m = (a^2 - b^2)^{1/2}, \qquad \gamma = \tfrac{1}{2} \log \frac{a + b}{a - b};$$

when m and γ are thus determined,

$$me^{\gamma} = a + b, \qquad me^{-\gamma} = a - b,$$

and the relations (5) are fulfilled. Then

$$m^2 \cosh 2\gamma = \tfrac{1}{2}m^2(e^{2\gamma} + e^{-2\gamma}) = \tfrac{1}{2}[(a + b)^2 + (a - b)^2]$$
$$= a^2 + b^2, \qquad m^2 \sinh 2\gamma = 2ab,$$
$$m^3 \cosh 3\gamma = a^3 + 3ab^2, \qquad m^3 \sinh 3\gamma = 3a^2b + b^3,$$
$$m^4 \cosh 4\gamma = a^4 + 6a^2b^2 + b^4, \qquad m^4 \sinh 4\gamma = 4a^3b + 4ab^3,$$

etc., so that the left-hand members of the identities involving the U's in (3) have on the ellipse the form $m^n \cosh n\gamma \cos n\theta$, while the left members involving the V's take the form $m^n \sinh n\gamma \sin n\theta$.

From (4) and (5),

$$m \cosh \gamma \cos \theta = a \cos \theta = r \cos \phi, \qquad m \sinh \gamma \sin \theta = r \sin \phi.$$

Hence

$$m(\cosh \gamma \cos \theta + i \sinh \gamma \sin \theta) = r(\cos \phi + i \sin \phi) = re^{i\phi}.$$

Let

$$b \cos \theta = r' \cos \psi, \qquad a \sin \theta = r' \sin \psi.$$

Then

$$m \sinh \gamma \cos \theta = r' \cos \psi, \qquad m \cosh \gamma \sin \theta = r' \sin \psi,$$
$$m(\sinh \gamma \cos \theta + i \cosh \gamma \sin \theta) = r'e^{i\psi}.$$

From these relations it follows that

$$re^{i\phi} + r'e^{i\psi} = m(\cosh\gamma + \sinh\gamma)(\cos\theta + i\sin\theta) = me^{\gamma}e^{i\theta},$$
$$re^{i\phi} - r'e^{i\psi} = me^{-\gamma}e^{-i\theta}. \tag{6}$$

Consequently

$$m^n(e^{n\gamma}e^{ni\theta} + e^{-n\gamma}e^{-ni\theta}) = (re^{i\phi} + r'e^{i\psi})^n + (re^{i\phi} - r'e^{i\psi})^n. \tag{7}$$

The real and pure imaginary parts of the left-hand member are respectively

$$m^n(e^{n\gamma} + e^{-n\gamma})\cos n\theta = 2m^n\cosh n\gamma\cos n\theta$$

and

$$im^n(e^{n\gamma} - e^{-n\gamma})\sin n\theta = 2im^n\sinh n\gamma\sin n\theta,$$

so that these quantities can be evaluated by separating real and imaginary components on the right of (7). On the other hand, multiplication of the two equations (6) by each other gives

$$r^2e^{2i\phi} - r'^2e^{2i\psi} = m^2. \tag{8}$$

The rest of the calculation is probably easier to follow on a first reading if carried through, not for arbitrary n, but for a particular value of n small enough to permit writing the formulas at full length, and at the same time large enough to illustrate the essential points involved. With $n=6$, one-half the right-hand member of (7) is

$$r^6e^{6i\phi} + 15r^4r'^2e^{i(4\phi+2\psi)} + 15r^2r'^4e^{i(2\phi+4\psi)} + r'^6e^{6i\psi},$$

the terms with odd powers of r' in the binomial expansions cancelling each other. On replacement of $r'^2e^{2i\psi}$ by $r^2e^{2i\phi} - m^2$ from (8) this takes the form

$$r^6e^{6i\phi} + 15r^4e^{4i\phi}(r^2e^{2i\phi} - m^2) + 15r^2e^{2i\phi}(r^2e^{2i\phi} - m^2)^2 + (r^2e^{2i\phi} - m^2)^3, \tag{9}$$

which becomes after expansion and collection of terms

$$A_6r^6e^{6i\phi} + A_4m^2r^4e^{4i\phi} + A_2m^4r^2e^{2i\phi} + A_0m^6 \tag{10}$$

with numerical coefficients $A_6, \cdots, A_0$. It is clear from the structure of the formulas that the coefficients are the same as those which are obtained if $\cos 6\theta$ as the real part of $(\cos\theta + i\sin\theta)^6$ is written in the form

$$\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$
$$= \cos^6\theta + 15\cos^4\theta(\cos^2\theta - 1) + 15\cos^2\theta(\cos^2\theta - 1)^2 + (\cos^2\theta - 1)^3$$

and rearranged as a polynomial in $\cos\theta$,

$$\cos 6\theta = A_6\cos^6\theta + A_4\cos^4\theta + A_2\cos^2\theta + A_0.$$

The expression (9) can in fact be represented by $m^6C_6(re^{i\phi}/m)$, if $C_n(\cos\theta)$ is the polynomial giving $\cos n\theta$ in terms of $\cos\theta$.

Separation of reals and pure imaginaries in (10) then gives

$$m^6 \cosh 6\gamma \cos 6\theta = (a^6 + 15a^4b^2 + 15a^2b^4 + b^6) \cos 6\theta$$
$$= A_6 r^6 \cos 6\phi + A_4 m^2 r^4 \cos 4\phi + A_2 m^4 r^2 \cos 2\phi + A_0 m^6,$$
$$m^6 \sinh 6\gamma \sin 6\theta = (6a^5b + 20a^3b^3 + 6ab^5) \sin 6\theta$$
$$= A_6 r^6 \sin 6\phi + A_4 m^2 r^4 \sin 4\phi + A_2 m^4 r^2 \sin 2\phi$$

for points (r, ϕ) on the ellipse, that is to say,

$$(a^6 + 15a^4b^2 + 15a^2b^4 + b^6)U_6 = A_6 u_6 + A_4(a^2 - b^2)u_4 + A_2(a^2 - b^2)^2 u_2 + A_0(a^2 - b^2)^3,$$
$$(6a^5b + 20a^3b^3 + 6ab^5)V_6 = A_6 v_6 + A_4(a^2 - b^2)v_4 + A_2(a^2 - b^2)^2 v_2.$$

It is clear that the work that has been done in the last two paragraphs for $n=6$ is applicable in principle for an arbitrary value of n, requiring merely a repetition of the same steps with an appropriate notation for the general binomial coefficients. In tabulating as many of the U's and V's as may be desired for computational purposes, the polynomials in a and b which appear in the left-hand members of the identities can of course be written down at once from the powers of $(a+b)^n$. The coefficients corresponding to the A's, identified by the fact that they serve to express $\cos n\theta$ in terms of powers of $\cos \theta$, can be obtained rapidly in succession from the recurrence relation

$$\cos (n + 1)\theta = 2 \cos \theta \cos n\theta - \cos (n - 1)\theta.$$

The u's and v's can be found as polynomials in x and y from the expansion of $(x+iy)^k$, or can be left in the form $r^k \cos k\phi$, $r^k \sin k\phi$.

5. The boundary value problem for an ellipsoid. The method outlined above can be adapted in its earlier stages to the solution of the harmonic boundary value problem for an ellipsoid, though the formulas do not appear ultimately to be similarly manageable.

Let the ellipsoid, with the rectangular equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{11}$$

be represented parametrically by

$$x = a \cos \theta, \qquad y = b \sin \theta \cos \phi, \qquad z = c \sin \theta \sin \phi. \tag{12}$$

The assigned boundary values can be regarded as constituting a given function $f(\theta, \phi)$. Under suitable hypotheses $f(\theta, \phi)$ can be expanded in a uniformly convergent Laplace series in terms of the spherical harmonics

$$\cos n\phi \sin^n \theta P_m^{(n)}(\cos \theta), \qquad \sin n\phi \sin^n \theta P_m^{(n)}(\cos \theta), \tag{13}$$

$m=0, 1, 2, \cdots ; n=0, 1, \cdots, m$, where $P_m^{(n)}(\cos \theta)$ denotes the nth derivative of the Legendre polynomial $P_m(\cos \theta)$ with respect to $\cos \theta$ as argument.*

* For the form of the series see for example the Carus Monograph cited above, pp. 118–126.

For $m=0, 1, 2$ the functions (13) are

$$1, \quad \cos\theta, \quad \cos\phi \sin\theta, \quad \sin\phi \sin\theta,$$

$$\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2},\ 3\cos\phi\sin\theta\cos\theta,\ 3\sin\phi\sin\theta\cos\theta,\ 3\cos 2\phi\sin^2\theta,\ 3\sin 2\phi\sin^2\theta;$$

since $\sin n\phi = 0$ for $n=0$ just $2m+1$ functions are defined for each m. It is apparent by inspection that the expressions listed can be written by means of (12) in the alternative form

$$(14) \qquad \begin{matrix} 1, & x/a, & y/b, & z/c, \\ \tfrac{3}{2}(x/a)^2 - \tfrac{1}{2}, & 3xy/(ab), & 3xz/(ac), & 3[(y/b)^2 - (z/c)^2], \quad 6yz/(bc). \end{matrix}$$

It can be shown* that each of the functions (13) is expressible as a polynomial of the mth degree in the three variables $\cos\theta$, $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, and so by the virtue of (11) in x/a, y/b, z/c or in x, y, z.

By substitution of the value of z^2 from (11), any polynomial can be reduced as far as values on the ellipsoidal surface are concerned to a form in which no term contains a power of z above the first. There are just $2m+1$ monomials of the mth degree, and

$$1 + 3 + 5 + \cdots + (2m+1) = (m+1)^2$$

monomials of the mth or lower degree, which contain no power of z higher than the first. On the other hand,† there are just $2m+1$ linearly independent homogeneous harmonic polynomials of the mth degree, and $(m+1)^2$ independent homogeneous harmonic polynomials of the mth or lower degree, and these can be taken as representing in terms of x, y, z the functions

$$(15) \qquad \rho^m \cos n\omega \sin^n\psi P_m^{(n)}(\cos\psi), \qquad \rho^m \sin n\omega \sin^n\psi P_m^{(n)}(\cos\psi),$$

if (ρ, ψ, ω) are spherical coordinates of an arbitrary point of space, so that

$$x = \rho\cos\psi, \qquad y = \rho\sin\psi\cos\omega, \qquad z = \rho\sin\psi\sin\omega.$$

These harmonic polynomials are linearly independent not merely as functions of three variables but also as functions in two dimensions on the ellipsoidal surface, since a relation of linear dependence connecting them there would contradict the theorem of uniqueness for the boundary value problem.

Consequently any polynomial is expressible on the surface as a linear combination of harmonic polynomials, and this is true in particular of the polynomial representations of the functions (13). If the terms of the Laplace series for $f(\theta, \phi)$ are replaced by the corresponding harmonic polynomials, there is obtained a series of harmonic functions which converges uniformly to $f(\theta, \phi)$ on the surface, under suitable hypotheses, and so converges uniformly to the solu-

* See for example Carus Monograph, pp. 126–127; for the present purpose the factor ρ of the passage cited is to be set equal to 1, and x, y, z (from page 106 of the Monograph) are to be replaced by z, x, y.

† See Carus Monograph, pp. 126–129.

tion of the boundary value problem throughout the interior of the ellipsoid, by the same sort of application of the maximum-minimum theorem that was used in the next to the last paragraph of §2 for the ellipse.

The polynomials corresponding to (15) for $m=0, 1, 2$ are

$$(16)\qquad \begin{matrix} 1, & x, & y, & z, & \\ x^2-\tfrac{1}{2}y^2-\tfrac{1}{2}z^2, & 3xy, & 3xz, & 3(y^2-z^2), & 6yz. \end{matrix}$$

It can be readily verified that these satisfy Laplace's equation. Of the nine polynomials (14), all but in fifth and eighth are harmonic as they stand, being constant multiples of the corresponding members of the sequence (16). Each of the remaining two can be expressed with undetermined coefficients as a linear polynomial in $x^2-\frac{1}{2}y^2-\frac{1}{2}z^2$, $3(y^2-z^2)$, and $(x/a)^2+(y/b)^2+(z/c)^2$. By evaluation of the coefficients and substitution of the value 1 for $(x/a)^2+(y/b)^2+(z/c)^2$ it is found that on the ellipsoid

$$\tfrac{3}{2}(x/a)^2-\tfrac{1}{2}=(1/D)[A_1(x^2-\tfrac{1}{2}y^2-\tfrac{1}{2}z^2)+B_1(3y^2-3z^2)+C_1],$$
$$3[(y/b)^2-(z/c)^2]=(1/D)[A_2(x^2-\tfrac{1}{2}y^2-\tfrac{1}{2}z^2)+B_2(3y^2-3z^2)+C_2],$$

with

$$D=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2},\qquad A_1=\frac{3}{2a^2}\left(\frac{1}{b^2}+\frac{1}{c^2}\right),\qquad B_1=\frac{1}{4a^2}\left(\frac{1}{c^2}-\frac{1}{b^2}\right),$$
$$C_1=\frac{1}{a^2}-\frac{1}{2b^2}-\frac{1}{2c^2},$$
$$A_2=\frac{3}{a^2}\left(\frac{1}{c^2}-\frac{1}{b^2}\right),\qquad B_2=\frac{1}{2a^2b^2}+\frac{1}{2a^2c^2}+\frac{2}{b^2c^2},\qquad C_2=3\left(\frac{1}{b^2}-\frac{1}{c^2}\right).$$

It is clear that the work would have to be continued much further to bring to light any general law of formation of the coefficients.

NOTE ON THE t-TEST

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1. Definition. Given a normal universe whose mean α is assumed to be known but whose standard deviation σ is unknown, the hypothesis that a given sample of $n'=n+1$ elements with mean $\bar{x}$ and variance $s^2=\sum(x-\bar{x})^2/n$ may be considered as a random sample of such a universe at a certain level of probability P is accepted or rejected according as

$$(1)\qquad t=\frac{(\bar{x}-\alpha)\sqrt{n'}}{s}$$

has a value less or greater than that derived from

$$P = \frac{2\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)} \int_t^\infty \frac{dt}{\left(1+\frac{t^2}{n}\right)^{(n+1)/2}}. \tag{2}$$

In case n is very large the value of t for $P=.05$ is 1.96; in case the sample has only $n'=2$ elements so that n has its minimum value 1, t is 12.7.

2. Distribution of t for samples of 2. For this case of $n=1$ it is sometimes practicable to derive the distribution of t in (1) for universes other than normal. We have indeed

$$\bar{x} = \frac{x_1 + x_2}{2}, \qquad s = \frac{|x_1 - x_2|}{\sqrt{2}}. \tag{3}$$

If $f(x)$ is the postulated probability distribution of the underlying universe, the probability of the observed sample (x_1, x_2) is the integral

$$\iint f(x_1)f(x_2)dx_1dx_2 \tag{4}$$

extended over all parts of the x_1x_2-plane which contain points (x_1, x_2). As the distribution in the plane is symmetrical, with respect to x_1 and x_2, we may use the integral

$$2\iint f(x_1)f(x_2)dx_1dx_2$$

extended over all parts of the x_1x_2-plane containing points (x_1, x_2) with $x_1 \geqq x_2$ and we may take $s=(x_1-x_2)/\sqrt{2}$. We may then solve for

$$x_1 = \bar{x} + \tfrac{1}{2}\sqrt{2}\,s, \qquad x_2 = \bar{x} - \tfrac{1}{2}\sqrt{2}\,s, \qquad dx_1dx_2 = \sqrt{2}\,d\bar{x}ds$$

and consider the integral

$$2\sqrt{2}\iint f(\bar{x} + \tfrac{1}{2}\sqrt{2}\,s)f(\bar{x} - \tfrac{1}{2}\sqrt{2}\,s)d\bar{x}ds \tag{5}$$

over all possible values of $\bar{x}$ and s. If we eliminate $\bar{x}$ by introducing from (1)

$$\bar{x} = \frac{st}{\sqrt{2}} + \alpha, \qquad d\bar{x} = \frac{s}{\sqrt{2}}dt \tag{6}$$

we have

$$2\iint f\left(\frac{st}{\sqrt{2}} + \alpha + \tfrac{1}{2}\sqrt{2}\,s\right)f\left(\frac{st}{\sqrt{2}} + \alpha - \tfrac{1}{2}\sqrt{2}\,s\right)sdsdt \tag{7}$$

extended over all effective parts of the ts-plane, and if the proper limits for s

in terms of t can be found and the integrations can be performed, the distribution of t may be obtained.

For the case of the J-type exponential distribution

$$f(x) = ae^{-ax}, \qquad x \geqq 0,$$

with a mean value $1/a$, the result is

$$\frac{2a^2}{e^2}\int\int e^{-\sqrt{2}as}t\,ds\,dt. \tag{8}$$

The region of integration in the x_1x_2-plane is that below the line $x_1=x_2$, $\bar{x}$ may have any non-negative value, s ranges from $s=0$ on that line to $s=\sqrt{2}\bar{x}$ on the x_1-axis, and for this value of s we have

$$s = \frac{\sqrt{2}/a}{1-t} \quad \text{since} \quad \alpha = 1/a. \tag{9}$$

The region of integration in the $\bar{x}s$-plane is that under the line $s=\sqrt{2}\bar{x}$. As t is constant along the lines issuing from the point $(\bar{x}=1/a,\ s=0)$ in the $\bar{x}s$-plane, the range of s is from 0 to ∞ for $t>1$ whereas for $t<1$ the range is from 0 to the value of s given by (9). The integration must therefore lead to two different expressions, which are:

$$\frac{1}{e^2t^2} \quad \text{if} \quad t>1 \quad \text{and} \quad \frac{1}{e^2t^2}\left(1-\frac{1+t}{1-t}e^{-2t/(1-t)}\right) \text{ if } t<1, \tag{10}$$

the total range of t being from $-\infty$ to $+\infty$.

The two functions have the common value e^{-2} at $t=1$. The distribution has a mode at $t=.42$ where it takes the value $2.41/e^2=.326$. At $t=0$ the value is $2/e^2=.271$. The integral for $t<1$ which vanishes at $t=-\infty$ is

$$1-\frac{1}{e^2t}+\frac{1}{e^2}\left(\frac{1}{t}-1\right)e^{-2t/(1-t)}$$

from which it may be ascertained that the median is at $t=-.40$. For the normal distribution for $n=1$ the t distribution is

$$\frac{1}{\pi}\,\frac{1}{1+t^2}$$

which is symmetrical with the mode at $t=2$ of value .318. This curve falls off as $.318/t^2$ for large values of t, whether positive or negative, whereas the distribution (10) falls off as $.135/t^2$ for large positive and as $1.135/t^2$ for large negative t and is very skew. Neither distribution has a mean because the integrand falls off only as $1/t^2$ at infinity.

3. Critical ratios for $P=.05$. If we should automatically take $t=12.7$ from the tables for t there would not be a chance of .025 for $t>12.7$ or for $t>-12.7$;

the chance for $t>12.7$ would actually be about .01 and for $t<-12.7$ would be about .08, making a total of .09 instead of .05. What values of t would reject the hypothesis at the .05 level? That would depend on how one might define "significant" values of t for the skew curve. If we were to take those values beyond which the tail of the distribution contained an area of .025 we should find the limits at $t=5.4$ and $t=-44$. If, however, we should set the limits of t by the condition that a line parallel to the axis should cut the distribution (10) in two points such that the total area in the two tails beyond those points should be .05 the positive value of t would be about 10, the negative value about -30.

It is not to be supposed that from any sample of 2 elements (or from any really small sample) one could tell the type of the universe from which the sample was drawn; if known at all, it would have to be as a result of previous experience with large samples or on the basis of adequate a priori theory. Furthermore if the universe were not normal presumably some test of significance other than the distribution of t defined by (1) should be used. Yet, as many persons will very naturally apply the t-test quite automatically, it is worth while to show students by elementary examples that if the universe is not normal there may be considerable differences between the probabilities determined from the tables for t and those calculated from various distributions. The reader can readily satisfy himself that if the universe were rectangular instead of normal, the .05 level would be for $t=19$ instead of 12.7.

THE ABEL-DINI AND ALLIED THEOREMS

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1. These observations are in the nature of a postscript to two papers in this MONTHLY: (i) T. H. Hildebrandt's paper *Remarks on the Abel-Dini theorem*, vol. 49 (1942), pp. 441–445, and (ii) my paper *Remarks on some generalizations of Cauchy's condensation and integral tests*, vol. 48 (1941), pp. 180–185. For the sake of brevity these two papers will be referred to as **H** and **R** respectively in what follows. Theorem I of **H** and its further extensions are all included in the third alternative of Theorem 1a of **R**. Further, Theorem 1a of **R**, in common with the Abel-Dini Theorem, can be derived from de la Vallée-Poussin's theorem quoted at the outset of **R**. This derivation of the Abel-Dini theorem is exactly like the derivation of its analogue for convergent series from another theorem of de la Vallée-Poussin which runs:*

THEOREM 1. *If* $m_n \downarrow 0$ *and* $F(x) \uparrow \infty$ *as* $x \to +0$, *then* $\sum(m_n-m_{n+1})F(m_n)$ *converges with* $\int_0 F(x)dx$ *and* $\sum(m_n-m_{n+1})F(m_{n+1})$ *diverges with* $\int_0 F(x)dx$.

To deduce from this the analogue of the Abel-Dini theorem we first note that $\sum(m_n-m_{n+1})/m_{n+1}$ is divergent. The divergence of $\sum(m_n-m_{n+1})/m_n$ now follows since, either $2m_{n+1}>m_n$ for all sufficiently large n which means

* Cours d'Analyse, vol. 1, 1926, p. 399, Ex. 5.

$(m_n-m_{n+1})/m_n>(m_n-m_{n+1})/2m_{n+1}$ for all such n, or $2m_{n+1}\leqq m_n$ for an infinity of n which means $(m_n-m_{n+1})/m_n\geqq\frac{1}{2}$ for these n. Hence we have in any case the divergence of $\Sigma(m_n-m_{n+1})/m_n^p$ for $p=1$ and consequently for $p>1$ also. Thus we are led to the

COROLLARY. *If* $m_n\downarrow 0$, $\sum(m_n-m_{n+1})/m_n^p$ *converges or diverges according as* $p<1$ *or* $p\geqq 1$.

Theorems related to this very much as Theorems I, II of **H** are related to the Abel-Dini theorem appear as particular cases of the following deduction from Theorem 1 when $F(x)=1/xl_1(1/x)\cdot l_2(1/x)\cdot\cdots[l_k(1/x)]^p$, where $l_k\xi=\log\log\cdots$ (k times)ξ.*

THEOREM 1a. *If, in Theorem* 1, *either* $\overline{\lim}\ (m_n^{-1}-m_{n-1}^{-1})<\infty$, *or* $\underline{\lim}\ (m_n-m_{n+1})/(m_{n-1}-m_n)>0$, *or* $\underline{\lim}\ m_n/m_{n-1}>0$, *then* $\sum(m_n-m_{n+1})F(m_{n+1})$ *converges or diverges with* $\int_0 F(x)dx$.

2. The treatment accorded above to the Abel-Dini theorem and to its analogue support a point of view which is explained in **R** and which may be briefly expressed thus. Our familiar convergence theorems as well as canonical forms for the criteria of convergence can be shown to have their origin in Maclaurin's method of condensing a series by means of an integral. This point of view helps us to achieve a pedagogically satisfactory "systematization of the general theory of convergence." For instance it enables us to bring Kummer's test within the scope of such a systematization more naturally than attempts like Knopp's.† Also it dispenses with a special argument like Bromwich's to explain why a *divergent* series should figure prominently in this criterion for convergence.‡ Furthermore it makes it clear that Kummer's test is only a criterion 'of the second kind' which corresponds to Pringsheim's generalization of Cauchy's root-test§ regarded as a criterion 'of the first kind.' These points, implicit in **R**, will now be made explicit with reference to the following statement of Kummer's and Pringsheim's criteria.

C. If $a_n>0$, $d_n>0$ $(n=1, 2, \cdots)$, *the latter are the terms of a divergent series and* $D_n=\sum_{\nu=1}^{n}d_\nu$, *then*

either (i)

$$\left(\frac{a_n}{d_n}\right)^{1/D_n}\leqq\theta<1 \qquad \textit{[PRINGSHEIM]},$$

or (ii)

$$\frac{1}{d_n}\left(\frac{a_{n+1}/d_{n+1}}{a_n/d_n}-1\right)\leqq-\rho<0 \qquad \textit{[KUMMER]},$$

ensures the convergence of $\sum a_n$.

* For a discussion of some allied problems see my paper, On certain theorems of Pringsheim, Tôhoku Math. Journ. 43, 1937, pp. 122–126.

† Knopp, Infinite Series, §42, in particular, pp. 310–311.

‡ Bromwich, Infinite Series (ed. 2), p. 38.

§ Math. Annalen, vol. 35, 1890, p. 340; or Knopp, *loc. cit.*, p. 309.

These results are obtained with $F(x)=e^{-\rho x}$, (i) in Theorem 1 of **R**, (ii) in the first alternative of Theorem 1a of **R**. For, (i) implies that

$$\frac{a_n}{d_n} \leqq e^{-\rho D_n} \qquad (e^{-\rho}=\theta,\ \rho>0)$$

(ii) has a similar implication since $0<\rho d_n<1$ and therefore

$$\frac{a_{n+1}/d_{n+1}}{a_n/d_n} \leqq 1-\rho d_n < e^{-\rho d_n}$$

whence

$$\frac{a_n/d_n}{a_1/d_1} < e^{-\rho D_{n-1}} = e^{-\rho D_n+\rho d_n} < e^{-\rho D_n+1}$$

or

$$\frac{a_n}{d_n} < Ke^{-\rho D_n} \qquad \left(K=\frac{a_1}{d_1}e\right).$$

Theoretically $\mathcal{C}$ is incomplete without a statement such as $\mathcal{D}$ which sets forth a criterion for the divergence of $\sum a_n$ in terms of a *convergent* series.

$\mathcal{D}$. *If* $a_n>0$, $c_n>0$ $(n=1, 2, \cdots)$, *the latter are the terms of a convergent series and* $R_n=\sum_{\nu=n+1}^{\infty}c_\nu$, *then*

either (i) $$\left(\frac{a_n}{c_n}\right)^{R_n} \geqq \lambda > 1,$$

or (ii) $$\left(\frac{1}{R_n^{-1}-R_{n-1}^{-1}}\right)\left(1-\frac{a_{n-1}/c_{n-1}}{a_n/c_n}\right) \geqq \rho > 0^*$$

ensures the divergence of Σa_n.

These results follow from Theorem 1 when we put $F(x)=e^{\rho/x}$, $m_n=R_n$.

3. A theorem, of much the same scope as Theorem I of **H**, and like it included in the third alternative of Theorem 1a of **R**, is:

THEOREM 2. *If* $D_n=\sum_{\nu=1}^{n}d_\nu$ $(d_\nu>0)\to\infty$ *and* $D_n/D_{n-1}=O(1)$ *as* $n\to\infty$, *then the series*

$$\sum \frac{d_n}{D_n^p}(\log D_n)^q,\dagger$$

$$\sum \frac{d_n}{D_n^p}\left(\frac{d_1}{D_1}+\frac{d_2}{D_2}+\cdots+\frac{d_n}{D_n}\right)^q,$$

are convergent for $p>1$, *or* $p=1$ *and* $q<-1$; *divergent otherwise.*

* The hypothesis in $\mathcal{D}$ (ii) is the analogue of the hypothesis in $\mathcal{C}$ (ii) in the form:

$$\left(\frac{1}{D_n-D_{n-1}}\right)\left(\frac{a_n/d_n}{a_{n-1}/d_{n-1}}-1\right) \leqq -\rho < 0.$$

† The case $d_n=1$ is given in Bromwich, *loc. cit.*, p. 460, Ex. 22.

The case $p>1$ of the second series in this theorem admits of a considerable generalization in the direction of certain theorems of Hardy.* The generalization may be stated as follows.

THEOREM 3. *If* $\{d_n\}$ *is as in Theorem* 2, $\{a_n\}$ *non-negative and such that*

$$A_n = a_1d_1 + a_2d_2 + \cdots + a_nd_n \to \infty \quad as \quad n \to \infty,$$

then

$$\sum \frac{d_n}{D_n^p} A_n^q \leqq K \sum \frac{d_n}{D_n^p} (a_nD_n)^q,$$

where $p>1$, $q \geqq 1$, $K=K(p, q)$ *depends on* p *and* q *only, and the series on the right-hand side is convergent.*†

Proof. For $p>1$,

$$\phi_n = \frac{d_n}{D_n^p} + \frac{d_{n+1}}{D_{n+1}^p} + \cdots < \int_{D_{n-1}}^{\infty} \frac{dx}{x^p} = \frac{D_{n-1}^{1-p}}{p-1} = O(D_n^{1-p}).$$

Taking $A_0=0$, we have

$$\begin{aligned}
\sum_1^N \frac{d_n}{D_n^p} A_n^q &= \sum_1^N (\phi_n - \phi_{n+1}) A_n^q \leqq \sum_1^N \phi_n (A_n^q - A_{n-1}^q) \\
&= \sum_1^N O\left(\frac{1}{D_n^{p-1}}\right)(A_n^q - A_{n-1}^q) \\
&\leqq K \sum_1^N \frac{A_n^{q-1} a_n d_n}{D_n^{p-1}} \\
&\leqq K \left\{ \sum_1^N \frac{d_n}{D_n^q} (a_nD_n)^q \right\}^{1/q} \left\{ \sum_1^N \frac{d_n}{D_n^p} A_n^q \right\}^{(q-1)/q}
\end{aligned}$$

where we assume $q>1$ and use Hölder's inequality. The theorem for $q>1$ follows at once from this. When $q=1$ the argument remains valid without any appeal to Hölder's inequality.

That the theorem is not necessarily true for $p \leqq 1$ can be verified as follows. For $p \leqq 1$, $q>0$ the series on the left-hand side of the conclusion of Theorem 3 is divergent since A_n^q increases with n. The series on the right-hand side may be convergent as in the case $a_n = 1/D_n \log D_n$, $p=1$, $q>1$.

Theorem 3 becomes trivial when $\sum a_nd_n$ is convergent in which case it may be replaced by

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* See for instance Hardy, Littlewood and Polya, Inequalities, p. 247, Theorem 332.

† This result with $d_n=1$, is stated and proved by Hardy and Littlewood in Journal für Math., vol. 157, 1927, pp. 143–145.

THEOREM 4. *If* $\{d_n\}$ *is as in Theorem* 2, $\{a_n\}$ *non-negative and such that* $\sum a_n d_n$ *is convergent, and if*

$$A_n = a_n d_n + a_{n+1} d_{n+1} + \cdots,$$

then, for $p<1$, $q \geqq 1$,

$$\sum_m^\infty \frac{d_n}{D_n^p} A_n^q \leqq K \sum_m^\infty \frac{d_n}{D_n^p} (a_n D_n)^q,$$

where $K = K(p, q)$, *and the series on the right-hand side is convergent.*

This theorem, like the previous, is established by an obvious generalization of the argument employed by Hardy and Littlewood, *loc. cit.*, to prove the case $d_n = 1$.

We see that the theorem is not necessarily true for $p \leqq 1$ from the example: $p=1$, $a_n = 1/D_n(\log D_n)^{1+\delta}$, $0<\delta<1/q$.

It may be observed that Theorems 3 and 4 invite comparison with Theorems 1.1, 1.2, 2.1, 2.2 of E. T. Copson's *Note on series of positive terms* in the Journal of the London Mathematical Society, vol. 3 (1928) pp. 49–51.

DIFFERENCE EQUATIONS IN AVERAGE VALUE PROBLEMS

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1. Introduction. Although there have been some very interesting applications* of difference equations to probability problems, there seems to have been no organized attempt to apply them to average (or mean) value problems. In fact, there have been relatively few problems stated which require the average number of trials necessary to produce a certain number of successes or the average value of a stochastic or chance variable. These average value problems which have been discussed usually have been solved by infinite series. The method of difference equations is often easier and more direct than the use of series even in the simpler problems and becomes practically indispensable in some of the more complicated ones.

A discussion of difference equations and the *general* solution of a linear difference equation has been included in §2; these results are not new, but it appears that they are not well known and are of sufficient importance to warrant their inclusion. The reader may prefer to read the section only hastily, or omit it, before proceeding with the succeeding sections, and refer back as occasion demands.

Section 3 contains representative theorems on average value problems which are well-known; their solutions are obtained here from difference equations.

* Uspensky, J. V., Introduction to Mathematical Probability, McGraw-Hill Book Company, Inc., New York & London, 1937, pp. 166–7.

Some new problems are discussed in §4, both for their intrinsic interest and as vehicles to illustrate methods of solving linear difference equations in one variable. One of these problems was proposed as an advanced problem in this MONTHLY. Although its solution was obtained from a difference equation in three or four ways it appears that direct methods of solving these equations are not familiar mathematical techniques. Uspensky gives many excellent applications of them to problems in probability, and his book contains a great deal of material not to be found in other books on the same subject in the English language.

The problem in (4.3) is unique in that Bayes' theorem is required to establish the difference equation which yields its solution.

2. Difference equations in one variable. An equation in x, $f(x)$, $f(x+1)$, $f(x+2)$, $\cdots$, $f(x+n)$, where x is an independent variable, is called a difference equation* in the unknown function $f(x)$. A difference equation in which both $f(x)$ and $f(x+n)$ appear, but not $f(x+n+m)$, $m=1, 2, 3, \cdots$, is said to be of *order n*.

The function $\Delta f(x)=f(x+1)-f(x)$ is called the *difference* of $f(x)$, and the difference of $\Delta f(x)$, namely,

$$\Delta^2 f(x) = f(x+2) - 2f(x+1) + f(x)$$

is called the *second difference* of $f(x)$; successive differences are defined similarly. These equations can be solved for $f(x+1), f(x+2), \cdots, f(x+n)$ and substituted in equation (2.11) below to produce an equation of the form

$$A_n(x)\Delta^n f(x) + A_{n-1}(x)\Delta^{n-1} f(x) + \cdots + A_0(x)f(x) = \phi(x).$$

It was this notation which suggested the name *difference equation*.

2.1. Linear difference equations. If each $f(x+k)$, $0 \leqq k \leqq n$, appears linearly then the equation is called linear. We shall consider only linear difference equations in one unknown function. The general one can be written as

$$(2.11) \qquad a_n(x)f(x+n) + a_{n-1}(x)f(x+n-1) + \cdots + a_0(x)f(x) = \phi(x),$$

where the coefficients $a_i(x)$ and $\phi(x)$ are known functions. In all the problems which we shall consider, these coefficient functions are rational. If $\phi(x)=0$, then equation (2.11) is called homogeneous. If $\phi \neq 0$, then the equation

$$(2.12) \qquad a_n(x)f(x+n) + a_{n-1}(x)f(x+n-1) + \cdots + a_0(x)f(x) = 0$$

is called the *associated homogeneous equation* of (2.11).

We shall consider first the solution of a homogeneous linear difference equation since the solutions of the non-homogeneous equation can be derived from those of its associated homogeneous equation.

* Batchelder, P. M., An Introduction to Linear Difference Equations, Harvard University Press, 1927.

Boole, George, Treatise on the Calculus of Finite Differences, 1860.

If in any way we can obtain particular solutions $f_1(x), f_2(x), \cdots, f(x)$, of the homogeneous equation (2.12), then it follows by direct substitution that

$$f(x) = p_1(x)f_1(x) + p_2(x)f_2(x) + \cdots + p_n(x)f_n(x) \tag{2.13}$$

is also a solution, where the p's are arbitrary functions of period* one. In any problem where x assumes only the values $x=k+1, k+2, k+3, \cdots$, (where k is a constant) the difference equation will have a solution of the form

$$f(x) = c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x), \tag{2.14}$$

where the c's are constants. This follows since $p(x+1)\equiv p(x)$ has a constant value when x assumes any of the values $k+1, k+2, k+3, \cdots$.

The solutions (2.14) are not only solutions of equation (2.12), but are the only† solutions of the equation provided the particular solutions $f_1(x), f_2(x), \cdots, f_n(x)$ form a *fundamental system of solutions, i.e.*, are linearly independent. Then $f(x)$ is called the *general solution* of equation (2.12).

It is often possible to obtain a fundamental system of solutions of a linear homogeneous difference equation with constant coefficients, namely,

$$k_nf(x+n) + k_{n-1}f(x+n-1) + \cdots + k_0f(x) = 0,$$

of the form $f(x)\equiv\alpha^x$. When this substitution is made, the resulting *characteristic equation* of degree n,

$$k_n\alpha^n + k_{n-1}\alpha^{n-1} + \cdots + k_0 = 0 \tag{2.15}$$

determines n values, $\alpha_1, \alpha_2, \cdots, \alpha_n$ of α. Then the functions

$$\overset{x}{\alpha_1}, \overset{x}{\alpha_2}, \cdots, \overset{x}{\alpha_n}$$

constitute a fundamental system of solutions provided no two values of the α_i's are equal.

When exactly r roots of the characteristic equation are equal, the difference equation has solutions of the form $\alpha^xu(x)$, where $u(x)$ is a polynomial of degree $(r-1)$; that is, the r functions $\alpha_1^x, \alpha_2^x, \cdots, \alpha_r^x$, where $\alpha_1=\alpha_2= \cdots =\alpha_r$, can be replaced by the r independent particular solutions

$$\overset{x}{\alpha_1}, x\overset{x}{\alpha_1}, \cdots, x^{r-1}\overset{x}{\alpha_1}. \tag{2.16}$$

The general solution of the non-homogeneous equation (2.11) can be obtained‡ from the general solution of its associated homogeneous equation pro-

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* *Definition. If $p(x+k)=p(x)$, then $p(x)$ is a periodic function of period k.*
The difference equation

$$p(x+1) - p(x) = 0$$

is satisfied by every function whose value at $x+1$ is equal to its value at x, that is, by every periodic function of period *one*. The simplest analytic functions of period *one* are constants.

† Batchelder, *loc. cit.*, pp. 6–10.

‡ Batchelder, *loc. cit.*, pp. 13–16.

vided some particular solution $F(x)$ of (2.11) can be obtained. Then, as in the case of a linear differential equation, the most general solution of (2.11) is the sum of the particular solution and the general solution of the associated homogeneous equation, namely,

$$f(x) = F(x) + c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x).$$

After the general solution of the difference equation has been found, sufficient initial or boundary conditions will be needed to impose upon the general solution to obtain the particular solution which satisfies the proposed problem. In some problems there will be sufficient obvious initial conditions, and in others they will have to be obtained in any way possible.

2.2. The solution of a particular difference equation of order one. The method just discussed will be applied to the solution of the difference equation

$$f(x) = f(x-1) + k. \tag{2.21}$$

In practice it is often more convenient to replace the symbol $f(x)$ by N_x; equation (2.21) then becomes

$$N_x = N_{x-1} + k. \tag{2.22}$$

When we substitute $N_x = \alpha^x$ in the associated homogeneous equation, the value of α is determined to be $\alpha = 1$ and the general solution is $N_x = c_1$. In order to find a particular solution of (2.22) let $N_x = Ax + B$. Then we must have

$$Ax + B \equiv Ax - A + B + k.$$

Then $A = k$ and since the value of B is arbitrary, let $B = 0$. The general solution of (2.22) is then

$$N_x = c_1 + kx.$$

This difference equation is established in sections (3.3)–(3.6) and in each case a known boundary condition is that $N_0 = 0$. When this condition is imposed it follows that $c_1 = 0$ and the solution is

$$N_x = kx. \tag{2.23}$$

The procedures outlined in this section will be illustrated and utilized in §§3, 4.

3. Familiar average value problems and their solutions by difference equations. The following average value problems can be found in texts by Coolidge,* Uspensky, and other authors. With the exception of a few,† this list is complete. Different writers have obtained the solutions of these problems; often the solutions have been obtained by series and sometimes by applications of propositions (3.1) and (3.2), which we shall not prove.

* Coolidge, J. L., An Introduction to Mathematical Probability, Oxford, 1925, p. 28.

† See Uspensky, *loc. cit.*, Chapter IX, for example.

(3.1) *Proposition 1. The mathematical expectation (the average value) of the sum of several variables is equal to the sum of their expectations.*

(3.2) *Proposition 2. The mathematical expectation of the product xy of two independent variables x and y is equal to the product of their expectations.*

We shall illustrate the application of difference equations to the solution of several of these same problems. The method is essentially the same for all of them and avoids the necessity of devising ingeneous methods for the problems individually.

(3.3) *Proposition 3. If the probability of success on an individual trial is p, the average number of trials necessary to produce r successes is r/p.*

Solution: Let $f(r)=N_r$ be the average number of trials required to produce exactly r successes. Then after the first trial the number of successes yet to be obtained will be either $(r-1)$ or r with probabilities p and q, respectively, and we have the following difference equation:

$$N_r = p(1 + N_{r-1}) + q(1 + N_r), \quad \text{or}$$
$$N_r = N_{r-1} + 1/p.$$

Since the average number of trials to produce zero successes is equal to zero, then $N_0=0$ and from (2.23)

$$N_r = r/p.$$

(3.4) *Proposition 4. The average of the number of successes in n trials with constant probability p is np.*

Solution: Let N_n be the average number of successes in n trials. After the first trial, the total number of successes will be either: (a) one more than the number of successes in the remaining $(n-1)$ trials provided the first trial was a success, or (b) just the number of successes in the remaining $(n-1)$ trials provided the first trial was a failure. Then,

$$N_n = p(1 + N_{n-1}) + qN_{n-1},$$

or,

$$N_n = N_{n-1} + p.$$

Again, $N_0=0$ and from (2.23) it follows that

$$N_n = np.$$

(3.5) *Proposition 5. The average value of the discrepancy $d=r-np$ from the most likely number of successes in n trials of an event (where p is the probability of success on each trial) is zero.*

Solution: The discrepancy $d=r-np$ in n trials is equal to the sum of the discrepancies d_1 and d_2 in the first n_1 and last n_2 trials as is evident from the following equation:

$$d = r - np = (r_1 + r_2) - (n_1 + n_2)p = d_1 + d_2.$$

Let N_x be the average value of the discrepancy in x trials. Then by arbitrarily selecting $n_1=1$ and $n_2=n-1$, it follows from (3.1) that

$$N_n = N_1 + N_{n-1}.$$

But

$$N_1 = p(1-p) + q(0-p) = 0.$$

Then

$$N_n = N_{n-1} \quad \text{and since} \quad N_1 = 0, \quad \text{the solution is}$$
$$N_n = 0.$$

(3.6) *Proposition 6. The average value of the square of the discrepancy is npq.*

Solution: The square of the discrepancy d in n trials is not equal to the sum of the squares of the discrepancies d_1, d_2, but it does satisfy the following relation;

$$d^2 = d_1^2 + d_2^2 + 2d_1d_2. \tag{3.61}$$

Let N_x be the average value of the square of the discrepancy in x trials, and let $n_1=1$ and $n_2=n-1$. It follows from (3.61), (3.1) and (3.2) that;

$$N_n = N_1 + N_{n-1}.$$

Since

$$N_1 = p(1-p)^2 + q(-p)^2 = pq,$$

then

$$N_n = N_{n-1} + pq.$$

Again, since $N_0=0$, we have from (2.23) that

$$N_n = npq.$$

The solutions by difference equations of these two preceding theorems seem to be more direct than any of the orthodox methods which appear in books on statistics and probability.

4. Some new mean value problems. Although difference equations have not been used previously to solve the problems of the preceding section, solutions of the following ones become either impossible or very difficult without recourse to them.

(4.1) *Proposition 7.* Mr. A has x coins and Mr. B has y. They match coins until one player has won all the coins. What is the average number of tosses required to end the game?*

Solution: Let N_x be the average number of tosses required to end the game from the time Mr. A has x coins and Mr. B has y coins. Since after the first toss,

* This MONTHLY, Vol. 47, No. 5, pp. 324–5.

Mr. A will have either $x-1$ or $x+1$ coins with equal probabilities, the following difference equation can be deduced:

$$N_x = (1 + N_{x-1})/2 + (1 + N_{x+1})/2,$$
$$2N_x = N_{x-1} + N_{x+1} + 2. \tag{4.11}$$

This difference equation of order 2, with the obvious boundary conditions $N_0 = N_{x+y} = 0$, are sufficient to solve the problem.

As outlined in (2.1), the general procedure to find particular solutions of the associated homogeneous equation is to make the substitution

$$N_x = \alpha^x.$$

Then α must satisfy the characteristic equation

$$2\alpha = 1 + \alpha^2,$$

and $\alpha = 1, 1$. Since the roots are equal it follows from (2.16) that two particular solutions are 1 and x, and the general solution of the associated homogeneous equation is

$$N_x = A + Bx.$$

A particular solution of the non-homogeneous equation can often be found by substituting in it a polynominal with unknown coefficients of appropriate degree. If we substitute $N_x = Rx^2 + Sx + T$ in (4.11), it follows that

$$N_x = -x^2$$

is a particular solution of (4.11). Then the general solution of (4.11) is

$$N_x = -x^2 + Bx + A.$$

The obvious boundary conditions, $N_0 = 0$ and $N_{x+y} = 0$ are sufficient to determine A and B, and the solution is

$$N_x = -x^2 + (x + y)x = xy. \tag{4.12}$$

Besides illustrating a method of solving a difference equation which is frequently applicable, the result itself is rather unexpected and interesting. It is unexpected to find that, when $x = 1$, $y = 20$, the average number of tosses is as large as 20 when only *one* toss is required in about half the trials.

Although the notation N_x may suggest that player A is in some way singled out for attention, this is not the case, and N_x is the average number of tosses required to end the game by eliminating either player.

(4.2) *Proposition 8.* When the probabilities in the preceding proposition are equal to p and $q (p \neq q)$ on each individual trial, a solution becomes more difficult and the form more complicated. In this case, the difference equation which corresponds to (4.11) is the following:

$$N_x = pN_{x+1} + qN_{x-1} + 1. \tag{4.21}$$

The characteristic equation of the associated homogeneous equation is

$$\alpha = p\alpha^2 + q,$$

and the roots are

$$\alpha = 1, \quad q/p.$$

Equation (4.21) has a particular solution of the form $N_x = bx$ where b is determined by the equation $bx = pb(x+1) + qb(x-1) + 1$,
whence

$$b = (q - p)^{-1}.$$

Then the general solution of (4.21) is

$$N_x = R + S(q/p)^x + x(q - p)^{-1}.$$

Again it is obvious that $N_0 = N_{x+y} = 0$, and these conditions are sufficient to determine the following solution of the problem:

$$N_x = \frac{xq^{x+y} - (x + y)q^x p^y + yp^{x+y}}{(p - q)(p^{x+y} - q^{x+y})}. \tag{4.22}$$

Since *Proposition 8* reduces to *Proposition 7* when $p = q$, we should expect, that solution (4.22) would reduce to (4.12) when $p = q$. The function (4.22) is then indeterminate and of the form 0/0; it is not difficult to show that its limiting value as $p \to q \to \frac{1}{2}$ is $N_x = xy$.

(4.3) *Proposition 9.* In the matching game of (4.1), sometimes it will be player A who is eliminated and sometimes it will be player B. *What is the average number of tosses A_x required to eliminate player A?*

The a priori probabilities $\bar{p}_x$ and $\bar{p}_y$ that A (who has x coins) or B (who has y coins) will be eliminated* are:

$$\bar{p}_x = y/(x + y), \quad \bar{p}_y = x/(x + y).$$

The following difference equation can be established on the basis of Bayes' theorem:

$$A_x = \frac{1}{2}\left[\frac{y+1}{x+y}(1 + A_{x-1}) + \frac{y-1}{x+y}(1 + A_{x+1})\right] \div \frac{y}{x+y}, \quad \text{or} \tag{4.31}$$

$$(y + 1)A_{x-1} = 2yA_x - (y - 1)A_{x+1} - 2y.$$

The characteristic equation of the associated homogeneous equation is

$$(y + 1) = 2y\alpha - (y - 1)\alpha^2,$$

and,

$$\alpha = 1, \quad (y + 1)/(y - 1).$$

Then $A_x = C$ is a solution of the homogeneous equation, but $D(y+1)^x/(y-1)^x$

* Whitworth, W. A., Choice and Chance, G. E. Stechert & Co., New York, Reprint of Fifth Edition, p. 231.

cannot be expected to be a solution, nor is it, since $(y+1)/(y-1)$ is not a constant. It can be shown, however, that x/y is a second solution.

Then the general solution of the homogeneous equation is

$$A_x = C + D\left(\frac{x}{y}\right)^x.$$

A particular solution of (4.31) can be obtained by substituting $A_x = Rx^2 + Sxy + Ty^2$, and is

$$A_x = x(x + 2y)/3.$$

The general solution of (4.31) is then

$$A_x = C + D\left(\frac{x}{y}\right)^x + x(x + 2y)/3.$$

One obvious boundary condition is $A_0 = 0$. In this problem, A_{x+y} is meaningless but $A_1 = 1$ when $x = y = 1$, since exactly one toss is then required to eliminate either A or B. These two boundary conditions are sufficient to determine that $C = D = 0$, and the solution of the problem is

$$A_x = x(x + 2y)/3.$$

5. Conclusion.* Difference equations can be applied to many problems in summation of series, determination of coefficients in a series solution of a differential equation, in replacing the method of mathematical induction, and in other ways. Application of them is often more direct than the standard procedures.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

REVERSION OF SERIES WITH APPLICATIONS

J. B. REYNOLDS, Lehigh University

It has been the experience of the author that men doing research work in applied mathematics are frequently baffled by a difficulty that could be resolved by the method of reversion of series. It is his conviction that every student who is trained for engineering or other scientific work should be taught

* It is my pleasure to express here an appreciation of the interest and suggestions of H. M. Gehman in connection with this paper.

this subject at some time in his courses in mathematics. Texts on algebra usually present this subject in connection with the theory of undetermined coefficients but it will be presented here as a topic in differential calculus. This approach has certain advantages over the algebraic method. Applications will be given to show how the method may be used.

Let $y=f(x)$ be such that $f(0)=0$ and $f(x)$ is expansible in a series in powers of x; then by Maclaurin's theorem.

$$y = px + qx^2/2! + rx^3/3! + sx^4/4! + tx^5/5! + \cdots, \qquad (p \neq 0), \tag{1}$$

in which p, q, r, s, and t stand respectively for the values of the first, second, third, fourth, and fifth derivatives of y with respect to x, each evaluated for $x=0$.

If this series be reverted we may write

$$x = Py + Qy^2/2! + Ry^3/3! + Sy^4/4! + Ty^5/5! + \cdots \tag{2}$$

in which P, Q, R, S, and T stand respectively for the first, second, third, fourth, and fifth derivatives of x with respect to y evaluated for $x=0$ for which value y is, also, zero.

From the relation $dy/dx=1/(dx/dy)$, we find by successive differentiations the following relations

$$\begin{aligned} &P = 1/p, \qquad Q = -q/p^3, \qquad R = (3q^2 - pr)/p^5. \\ &S = (10pqr - 15q^3 - p^2s)/p^7, \\ &T = (15p^2qs + 10p^2r^2 - 105pq^2r + 105q^4 - p^3t)/p^9. \end{aligned} \tag{3}$$

These relations determine the values of the coefficients of the powers of y in the reverted series (2).

One advantage of using the calculus approach is that $y=f(x)$ does not have to be expanded into a series in order to obtain the reverted series giving the value of x in a power series in y as it does in the method of undetermined coefficients.

APPLICATIONS

1. Revert the series

$$y = x + x^2 + x^3/2 + x^4/6 + x^5/24 + \cdots$$

and find the value of x that satisfies $10xe^x=1$. From the successive derivatives of y with respect to x, evaluated for $x=0$, we find the values $p=1$, $q=2$, $r=3$, $s=4$, $t=5$; and by means of equations (3) $P=1$, $Q=-2$, $R=9$, $S=-64$, $T=625$; so that by equation (2)

$$x = y - y^2 + 3y^3/2 - 8y^4/3 + 125y^5/24 \cdots$$

which is the required reverted series. Since the given series is the expansion of xe^x, the value of y that makes x satisfy $10xe^x=1$ is 0.1. This value put into the reverted series gives $x=0.09128$.

2. Compute the root of $x^3+10x^2+8x-120=0$ that lies between 2 and 3. In order to have the reverted series converge as rapidly as possible we reduce the roots by 3 rather than by 2 and get the equation $x^3+19x^2+95x+21=0$. If now we let

$$y = x^3 + 19x^2 + 95x$$

so that $y=0$ for $x=0$, the equation will be satisfied when $y=-21$. From the assumed value of y we find

$$p = 95, \qquad q = 38, \qquad r = 6, \qquad s = 0;$$

whence

$$P = 1/95, \quad Q = -38/(95)^3, \quad R = 3762/(95)^5, \quad S = -606480/(95)^7.$$

These values give by equation (2)

$$x = y/95 - (1/5)(y/95)^2 + (33/475)(y/95)^3 - (14/475)(y/95)^4 \cdots$$

and for $y=-21$, $x=-0.23165$; so that the corresponding root of the original equation is $3-0.23165=2.76835$.

3. Solve the equation

$$x + e^x + \sin x + \tan x = 2.$$

Let

$$y = x + e^x + \sin x + \tan x - 1,$$

so that $y=0$ for $x=0$; then the given equation will be satisfied when $y=1$. From the assumed value of y we find

$$p = 4, \qquad q = 1, \qquad r = 2, \qquad s = 1, \qquad t = 18;$$

whence

$$P = 1/4, \quad Q = -1/64, \quad R = -5/1024, \quad S = 49/(4)^7, \quad T = -1007/(4)^9.$$

When these values are substituted into equation (2) we get

$$x = y/4 - 2(y/16)^2 - (10/3)(y/16)^3 + (49/6)(y/16)^4 - (1007/30)(y/16)^5 \cdots .$$

When y is set equal to 1 in this, we find $x=0.24147$ which is the required root.

THE ADJOINT OF A LINEAR DIFFERENTIAL EQUATION

L. H. McFARLAN, University of Washington

Let the expression $L(y)$ be defined by

$$L(y) = a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny, \tag{1}$$

where the a's are functions of x. Associated with such an expression is another form,

$$M(z) = b_0z^{(n)} + b_1z^{(n-1)} + \cdots + b_nz, \tag{2}$$

known as the adjoint of $L(y)$. The b's are functions of x which are expressible in terms of the a's and certain of their derivatives, which are assumed to exist. The expressions $L(y)$ and $M(z)$ satisfy a relation of the form

$$zL(y) - yM(z) = dR/dx \tag{3}$$

where R is a bilinear form with coefficients expressible in terms of the a's and certain of their derivatives and is linear in each of the two sets of functions $(y, y', \cdots, y^{(n-1)})$ and $(z, z', \cdots, z^{(n-1)})$. The differential equation $M(z)=0$ is said to be the adjoint differential equation of $L(y)=0$ and the relation is a symmetrical one.

A simple method of obtaining the form $M(z)$ from $L(y)$ is given by using one of the forms of the Euler equation of the calculus of variations. The left hand member of the expression (3) will be designated by ψ. It will be assumed that the function $z(x)$ has been assigned and that the form $M(z)$ is determined but is, so far, unknown. Then ψ may be considered as a function of the set of variables $(x, y, y', \cdots, y^{(n)})$ and is an exact differential of the expression R. Hence if one were to consider the problem of minimizing the integral

$$I = \int_{x_0}^{x_1} \psi dx \tag{4}$$

the values of I would depend only upon the sets of initial values of $(x, y, y', \cdots, y^{(n-1)})$ at x_0 and x_1 and not on the particular form of the function $y(x)$. A necessary and sufficient condition for this to be the case is that the Euler differential equation for such an integral

$$\psi_y - \psi'_{y'} + \psi''_{y''} + \cdots + (-1)^n \psi^{(n)}_{y^{(n)}} = 0 \tag{5}$$

vanishes identically.

For the particular form of ψ as defined above, the expression (5) is seen to be

$$a_n z - M(z) - (a_{n-1}z)' + (a_{n-2}z)'' + \cdots + (-1)^n (a_0 z)^{(n)} = 0, \tag{6}$$

which immediately gives the expression

$$M(z) = (-1)^n (a_0 z)^{(n)} + (-1)^{n-1} (a_1 z)^{(n-1)} + \cdots + a_n z.$$

AN ACKNOWLEDGEMENT

H. V. Craig, University of Texas

With regard to my recent comment in the June–July Monthly concerning the development of the formulas for $D \sin x$ and $D \cos x$ by vector methods, I learn from Professor Louis Brand that he presented essentially the same observation in his book *Vectorial Mechanics* (1930), example 1, page 178. The basic facts are so well known and the argument so obvious I suspected that the method had been recorded by some of the earlier workers in quaternions and vector analysis. A time consuming, although obviously insufficient, search of the literature was fruitless. I regret that I failed to find Professor Brand's example.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1943–44

Mathematics Club, Oberlin College

Semi-monthly meetings were held at which members of the club presented papers:

Why study mathematics, by Professor C. H. Yeaton

Geometric constructions with paper folding, by Rodney Hood

Geometric constructions with compass only, by Sarah-Lou Lotz

Maps, by Alan Chaney

The game of Nim, by Elizabeth Frazier

Simple diophantine equations, by Norman Weinstein

Vectors, by George Ritchie

Finite geometries, by Dora Sherman: a talk based on the article *Four finite geometries*, by H. F. MacNeish, this MONTHLY, Jan. 42, p. 15

What is a proof?, by Thomas Hargrove

Distances between points of a Cantor set, by Professor J. F. Randolph

A new distance function, by Mary Kinsman

Aerial navigation constructions, by Robert Graves

Repeating decimals, by Elizabeth MacKay

Algebraic equations, by Artha-Jean Burington

The major project of the club for the year was the writing and adoption of a constitution.

In addition to teas at the beginning of each meeting, the club sponsored a Christmas party, at which Professor Vance and Professor Rodabaugh debated the subject

Resolved that Pi(e) is more delectable than mathematical.

The club also sponsored a party honoring Mrs. C. H. Yeaton, who retired from the faculty this year. The last meeting of the year was a banquet in honor of Professor Mary E. Sinclair, head of the mathematics department, who also retires this year, after thirty-seven years of service to the college. The club is looking forward to its fiftieth anniversary next year.

Officers for the year were: President, Dora Sherman; Vice-President, George Ritchie; Secretary-Treasurer, Ruth Cheney; Program Chairman, Rodney Hood; Social Chairman, Mary Kinsman; Faculty Advisers, Professor Vance and Professor Rodabaugh. Officers elected for the year 1944–45 are: President, Ruth Cheney; Vice-President and Program Chairman, Rodney Hood; Secretary-Treasurer, Mary Kinsman; Social Chairman, Sarah-Lou Lotz; Publicity Chairman, Norman Weinstein; Faculty Adviser, Professor Vance.

Kappa Mu Epsilon, Mount St. Scholastica College

The theme for the year was *Mathematics as it is studied and taught in other countries*. Each monthly meeting was given over to a particular country or group of countries as follows:

FRANCE:

Sur les probabilitès, LaPlace, translated by Ann Hughes
La gêometrie, Descartes, treated by Patricia Warwick
Sophie Germain, by Mary Lou Maloney
Comparative study of mathematical textbooks, by Virginia Meyers.

SPANISH-SPEAKING COUNTRIES (open meeting with guest speakers):

Rural education and personal teaching experience in Mexico, by Mary Rachel Robleda from Mexico City
Student's view of Mexican education, by Fanny Mary Dorantes from Mexico City
Report on the nature of South American mathematical periodicals, by Mary Catherine Growney
Translation of a letter from Bernardo Baidaff, editor of *Boletin Mathematico*, by Katherine Zeller
Status of mathematics in the University of the Philippines, by Virginia Harrison.

BRITISH ISLES:

Brief history of the development of mathematics in England, by Margaret Mary Wolters.
London Royal Society, by Mary Frances Herrman
Newton-Leibnitz controversy, by Doris Beckman
Invariant twins—Sylvester and Cayley, by Mary Ann Wirtz.

GERMANY:

Life and work of Kepler, by Mary Jane Fox
Mathematical contributions of Gauss, by Mary Davis
Translation of a German work on the origin of counting, by Katherine Molloy.

The major social events were the Christmas party and the formal buffet supper following the initiation of new members. Noteworthy activities of the year were the U. S. War Stamp Drive and Mathematical Olympics. This latter was arranged in bazaar form featuring intellectual concessions such as *Mathematical Fortune-telling*; identifying pictures of mathematicians by clever clues; demonstration of Marchant Electric Calculator. The purpose of the festival was to determine the eligibility of prospective members. The program closed with the presentation of *Snow White and the Seven Dwarfs*.

For five consecutive years the honor of valedictorian has gone to a member of the Kansas Gamma Chapter—this year to Miss Virginia Meyers of Evanston, Illinois, chapter president, who also merited election to Kappa Gamma Pi, National Honorary Society. Kansas Gamma was also honored by the publication

in the spring issue of the *Pentagon*, of the research paper of one of its alumnae members, Muriel Thomas.

Officers-elect for the year 1944–45 are: President, Mary Lou Maloney; Vice-President, Mary Catherine Growney; Secretary, Katherine Zeller; Treasurer, Ann Hughes; Secretary Descartes and Faculty Sponsor, Sister Helen Sullivan, O.S.B.

Women's College Mathematics Club, University of Delaware

Meetings of the Club are always preceded by an informal tea, which is served to all who are interested in Mathematics on the campus. The five papers presented were:

Modern developments of some ancient ideas, by Dr. C. J. Rees

Foundations of mathematics, by Dr. Murray Mannos

An introduction to topology, by Dr. G. L. Walker

Must we use the base ten? by Dr. G. C. Webber

Opportunities for women with statistical training in market research, by Miss Elizabeth Watters. In this most interesting talk the opportunities were most vividly and clearly pointed out, and statistical procedure was explained and was used in illustrating work of this nature.

The Mathematics Club makes an annual contribution to the library of at least one book. The selection for the past year was *Aesthetic Measure*, by G. D. Birkhoff. Officers of the Club were: President, Agnes Wright; Vice-President, B. Catharine Carrick; Corresponding Secretary, Astrid Delitzsch; Treasurer, Jean Dukek; Chairman of Teas, Mary Edith Boyce.

Mathematics Club, Hunter College

Topics and speakers for the year 1943–44 were as follows:

Paths of heavenly bodies, by Mary Dolciani

Cryptography, by Marie Johnson

Origin of mathematical symbols, by Catherine Porcheddu

Excluded areas of algebraic curves, by Theo Gelbfarb

Asymptotes, by Melitta Lowy

The nine point circle, by Carol Podell

A few mathematical facts, by Augusta Schurrer and Gladys Heinlein

Pythagorean triplets, by Ilse Novak

Two theorems, by Professor Jewell Hughes Bushey

Pictorial topology, by Professor Saunders Mac Lane of Harvard University—an open meeting attended by about 175 members and guests.

The Club's two social events were the *Get Acquainted Tea* and the *Mad Hatter Tea Party*, financed by two successful cake sales. Officers for the year were: President, Theresa Danielson; Vice-President, Carol Podell; Secretary, Theo Gelbfarb; Treasurer, Mary Dolciani; Publicity Manager, Leila Rubashkin; Faculty Advisers, Mrs. Helen Kutman, Professor Marguerite Darkow.

Pi Mu Epsilon, Hunter College

The general topic considered for discussion by the Beta Chapter of *Pi Mu Epsilon* during the year 1943–44 was *Statistical theory and practice.* The fifteen papers presented were:

Probability functions, by Constance Heiden

Moment generating functions, by Mary Demitrack

Applications of the moment generating function to the normal probability function, by Lydia Blumenthal

The approach to normality of the Bernouilli and Poisson distributions, by Melitta Lowy

Semivariants, by Mildred Finger

Semivariants of the normal probability function, Poisson probability function, and the Type III probability function, by Gertrude Crosby

Semivariants and the distribution of sample means, by Marie Johnson

A method for the evaluation of a sum, by Gladys Rappaport

The moments of the sum or difference of two dependent variables, by Mae Perlstein

Some applications of the moments of the sum or difference of two variates, by Nelly Szabo

Fiducial probability and the Bernouilli distribution, by Regina Benbosset

Fiducial limits applied to the Poisson distribution, by Grace Leight

The theory of estimation, by Marie Lombardi

The method of maximum likelihood, by Miriam Josephson

The moments of a product, by Felicitas Reich.

Twenty-seven new members were initiated during the year. Prizes for best papers during the first and second semesters were copies of *Differential and Integral Calculus, v.2*, by Richard Courant, and *Mathematical Statistics*, by S. S. Wilkes, presented respectively to Marie Johnson and Lydia Blumenthal.

Officers for the current year were: Director, Dr. Leo Aroian; President, Pauline Palacek; Recording Secretary, Mary Dolciani; Corresponding Secretary, Gladys Heinlein; Treasurer, Melitta Lowy.

Mathematics Club, Case School of Applied Science

The following talks were presented at the meetings of the Club:

Supersonics, by J. T. Leiss

Some unintegrable integrals, by Professor P. E. Guenther: a most successful talk held at the home of Dean T. M. Focke

Mathematics of the electron, by Bernard Cohen

Geometry of the complex plane, by G. S. Springer

The density distribution of the stars, by Professor S. W. McCuskey of the Physics Department.

Officers of the Mathematics Club for the past season were: President, Bernard Cohen; Vice-President, George Springer; Secretary, Herbert Rutemiller. The Faculty Adviser was Professor Max Morris.

PROBLEMS AND SOLUTIONS

ELEMENTARY PROBLEMS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 646. *Proposed by Orrin Frink, Jr., Xenia, Ohio*

Prove that any two conjugate planes through a secant of a sphere meet the sphere in orthogonal circles.

E 647. *Proposed by V. Thébault, San Sebastián, Spain*

Let (m_1, m_2, m_3, m_4) be the barycentric coordinates of a point G with respect to a regular tetrahedron $A_1A_2A_3A_4$ of edge a (so that G is the centroid of masses m_1, m_2, m_3, m_4 at A_1, A_2, A_3, A_4). Obtain an expression for the distance A_4G.

E 648. *Proposed by Mary L. Boas and R. P. Boas, Jr., Tufts College and Harvard University*

Show that, when $x = 2\cos \pi/(n+1)$, the n-rowed determinant

$$\begin{vmatrix} x & 1 & 0 & 0 \cdots 0 & 0 \\ 1 & x & 1 & 0 \cdots 0 & 0 \\ 0 & 1 & x & 1 \cdots 0 & 0 \\ 0 & 0 & 1 & x \cdots 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \cdots x & 1 \\ 0 & 0 & 0 & 0 \cdots 1 & x \end{vmatrix}$$

has the value zero.

E 649. *Proposed by L. A. Santaló, Rosario, Argentine Republic*

A set of parallel line segments will be called "linear" if all of them can be cut by one straight line. Show by an example that an infinite set of parallel segments in one plane may have the property that every subset of three is linear while the whole set is not linear. (The segments are "open": not including their end points.)

E 650. *Proposed by Lloyd Dulmage, University of Manitoba*

If we first arrange n letters in a row, in a definite order, and then arrange below these letters

(*a*) a second row containing the same n letters so that no letter is repeated in any column, the number of possible arrays is ${}_2K_n$;

(*b*) a second row containing p of the letters of the first row together with $n-p$ other letters, so that no letter is repeated in any column, then the number of arrays is ${}_2K_{n,p}$;

(*c*) a second row containing a definite p of the letters of the first row (and $n-p$ empty spaces) so that exactly some q of these p letters do not appear below any of the p chosen letters (but the remaining $p-q$ letters appear below some $p-q$ of the p chosen letters), no letter being repeated in any column, then the number of arrays is ${}_2K_{n,p,q}$;

(*d*) two rows (second and third) each containing the same n letters, so that no letter is repeated in any column, then the number of arrays is ${}_3K_n$.

Show that the functions so defined satisfy the following relations:

$$(1) \qquad {}_2K_{n,p} = \sum_{r=0}^{p} (-1)^r \binom{p}{r} (n-r)!,$$

$$(2) \qquad {}_2K_{n,p,q} = \binom{p}{q}\binom{n-p}{q} {}_2K_{p,p-q},$$

$$(3) \qquad {}_2K_n = \sum_{q=0}^{\min(p,n-p)} {}_2K_{n,p,q} \cdot {}_2K_{n-p,n-p-q}$$

(for each value of p from 0 to n),

$$(4) \qquad {}_3K_n = \sum_{p=0}^{n} \sum_{q=0}^{\min(p,n-p)} (-1)^p \binom{n}{p} {}_2K_{n,p,q} ({}_2K_{n-p,n-p-q})^2.$$

SOLUTIONS

Arc and Area of an Epicycloid

E 600 [1943, 634]. *Proposed by J. H. Butchart, Grinnell College*

If the radii of the fixed and rolling circles are a and b respectively, the length of one arch of an epicycloid is $8(a+b)b/a$, and the area bounded by one arch and the fixed circle is

$$\pi(3a^2 + 8ab + 4b^2)b^2/a(a + 2b).$$

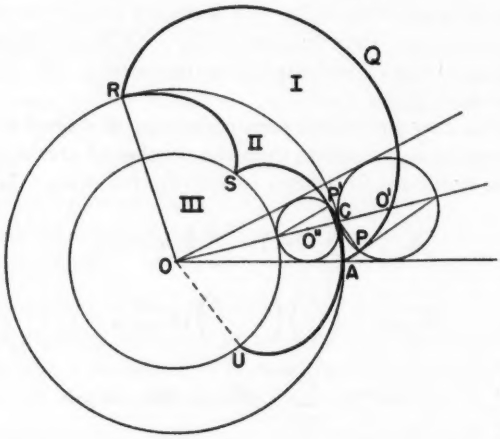
Corresponding formulas for the hypocycloid are obtained by changing the sign of b. Prove these results synthetically.

Solution by the Proposer. The formulas will be proved for the epicycloid $APQR$ generated by rolling a circle (O', b) on a circle (O, a). Modifications for the hypocycloid are obvious. Consider a circle $(O'', ab/(a+2b))$ tangent to (O') and homothetic to it with respect to the center O, and let it roll on an inner circle centered at O. As the point P, fixed on (O'), describes the epicycloid, the point P', fixed on (O''), describes the evolute. This evolute, an epicycloid whose arch subtends the angle $2\pi b/a$, is clearly similar to the original epicycloid, the

ratio of similitude being $a:(a+2b)$. Half an arch of the evolute, being equal to SQ, is

$$\frac{2ab}{a+2b}+2b=\frac{4ab+ab^2}{a+2b}.$$

Hence the entire arch of the original epicycloid is $8(a+b)b/a$.



To determine the area denoted by I, we find three linear relations connecting I, II, and III. Since III is equivalent to $OUAS$, we get

$$(\mathrm{I}+\mathrm{II}+\mathrm{III})/\mathrm{III}=(a+2b)^2/a^2.$$

Also

$$\mathrm{II}+\mathrm{III}=\pi ab.$$

A third relation is found by noting that $P'C/CP=a/(a+2b)$. An element of area of the region II+I can consist, to within infinitesimals of higher order, of an infinitesimal triangle formed by neighboring tangents to the evolute with base on the epicycloid. The arc of the fixed circle separating II from I can be replaced by a line parallel to the base of this triangle, dividing the sides in the ratio $a:(a+2b)$. The difference which this change makes is an infinitesimal of higher order than the small triangle itself. From this argument we see that

$$\mathrm{II}/(\mathrm{II}+\mathrm{I})=a^2/4(a+b)^2.$$

These three relations can be written as follows:

$$a^2\mathrm{I}+a^2\mathrm{II}-(4ab+4b^2)\mathrm{III}=0,$$

$$\mathrm{II}+\mathrm{III}=\pi ab,$$

$$a^2\mathrm{I}-(3a^2+8ab+4b^2)\mathrm{II}=0.$$

Solving for I, we get the announced result.

Incidentally, the same diagram gives the radius of curvature very directly as

$$\frac{4(a+b)b}{a+2b}\sin\frac{\phi}{2},$$

where ϕ is the angle $OO'P$.

A Second Meigs Hall Problem

E 611 [1944, 162]. *Proposed by D. F. Barrow, University of Georgia*

Suppose n students be standing an examination in a row of seats with an aisle at either end. If they finish in random order find the average number of disturbances caused by students passing over one another. (If a student passes over 3 others, he causes 3 disturbances. Assume that they go out so as to cause the smallest number of disturbances. Cf. E 531 [1943, 202 and 513].)

Solution by C. M. Feller, Providence, R. I. Suppose that, at a certain moment, there are k students left. If $k=2j$, the expected value of the number of disturbances caused by the first leaving student is

$$\frac{2}{k}\{0+1+2+\cdots+(j-1)\}=\frac{j-1}{2}=\frac{k}{4}-\frac{1}{2};$$

if $k=2j+1$, the expected value becomes

$$\frac{2}{k}\left\{0+1+2+\cdots+(j-1)+\frac{j}{2}\right\}=\frac{j^2}{k}=\frac{k}{4}-\frac{1}{2}+\frac{1}{4k}.$$

Summing over all k from 1 to n, we obtain the required average

$$\frac{n(n-3)}{8}+\frac{1}{4}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{m}\right),$$

where m is the greatest odd number not exceeding n.

Also solved by D. H. Browne, Irving Kaplansky, and the proposer. The proposer remarks that, in the case of several rows of students, we can find the average number of disturbances for each row and add the results. For instance, five rows of six students make an average of $13\frac{1}{3}$ disturbances, while six rows of five make only $9\frac{1}{3}$.

A Product of Four Consecutive Integers

E 612 [1944, 162]. *Proposed by V. Thébault, San Sebastián, Spain*

Show that $1110\cdot1111\cdot1112\cdot1113=1235431^2-1$ for any radix greater than 5. Generalize this result.

I. *Solution by E. D. Schell, Arlington, Virginia.* If 1_n is understood to mean a sequence of n digits 1 in the scale of r, let

$$R=r^n+r^{n-1}+\cdots+r+1\tfrac{1}{2}=1_{n+1}+\tfrac{1}{2}.$$

Then

$$R^2 = 123\cdots n(n+1)n\cdots 321 + 1_{n+1} + \tfrac{1}{4}$$
$$= 123\cdots n(n+2)(n+1)n\cdots 32 + \tfrac{1}{4}$$

and

$$1_n0\cdot 1_{n+1}\cdot 1_n2\cdot 1_n3 = (R-\tfrac{3}{2})(R-\tfrac{1}{2})(R+\tfrac{1}{2})(R+\tfrac{3}{2})$$
$$= R^4 - \tfrac{5}{2}R^2 + \tfrac{9}{16} = (R^2-\tfrac{5}{4})^2 - 1$$
$$= \{123\cdots n(n+2)(n+1)n\cdots 31\}^2 - 1$$

for any radix greater than $n+2$. The special case in the problem is when $n=3$.

II. *Solution by E. P. Starke, Rutgers University.* This is a special case of the algebraic identity

$$a(a+b)(a+2b)(a+3b) = (a^2+3ab+b^2)^2 - b^4,$$

in which we take $a=r^3+r^2+r$ (where r is the radix) and $b=1$, so that

$$a^2+3ab+b^2 = r^6+2r^5+3r^4+5r^3+4r^2+3r+1.$$

Since no coefficient exceeds 5, the identity is significant for any radix greater than 5. Other examples may be made up by taking other expressions for a; *e.g.*,

$$10100\cdot 10101\cdot 10102\cdot 10103 = 102040301^2 - 1 \quad \text{for} \quad r > 4,$$
$$12\cdot 13\cdot 14\cdot 15 = (17\epsilon)^2 - 1 \quad \text{for} \quad r > \epsilon = 11.$$

Also solved by H. N. Carleton, W. R. Talbot, J. A. Tierney and the proposer.

A Criterion for Unequal Cevians

E 613 [1944, 162]. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*
Can a triangle have two equal symmedians without being isosceles?

Solution by the Proposer.

LEMMA 1. *If the internal cevians AD and BE of the triangle ABC are such that $\sphericalangle BAD > \sphericalangle ABE$ and $\sphericalangle CAD > \sphericalangle CBE$, then $BE > AD$.*

Proof. Select a point F on CD so that $\sphericalangle FAD = \sphericalangle CBE$. Let BE meet AD at H, and AF at G. Since triangle FAB has a greater angle at A than at B, $FB > FA$. Since triangles FAD and FBG are similar, it follows that $AD < BG < BE$.

REMARK. If two cevians satisfy the conditions of this lemma, so do their isogonal conjugates.

LEMMA 2. *Two medians satisfy the conditions of Lemma* 1 *unless the triangle is isosceles.*

Proof. Let AD and BE be medians, and suppose that $BC > AC$. From the familiar formula for the median in terms of the sides it follows that $AD < BE$, whence $HB > HA$, and $\sphericalangle BAD > \sphericalangle ABE$. If CAD is obtuse the remaining condition is obviously satisfied. If not,

$$\frac{\sin \sphericalangle CAD}{\sin C} = \frac{CD}{AD} = \frac{BC}{2AD}, \qquad \frac{\sin \sphericalangle CBE}{\sin C} = \frac{CE}{BE} = \frac{AC}{2BE};$$

so

$$\frac{\sin \sphericalangle CAD}{\sin \sphericalangle CBE} = \frac{BC \cdot BE}{AC \cdot AD} > 1,$$

and

$$\sphericalangle CAD > \sphericalangle CBE,$$

as required.

From the two lemmas and the remark it follows that a triangle must be isosceles if two of its symmedians are equal. The famous Steiner-Lehmus theorem is another simple consequence of Lemma 1. Other special cases are of interest.

Also solved (by direct computation of the symmedians) by W. B. Carver, Howard Eves, R. A. Johnson, and Alan Wayne.

A Reducible Polynomial

E 614 [1944, 162]. *Proposed by Joseph Rosenbaum, Bloomfield, Conn.*

Prove that the polynomial $(x+y)^n-x^n-y^n$ is divisible by x^2+xy+y^2 when $n \equiv 5 \pmod 6$, and by $(x^2+xy+y^2)^2$ when $n \equiv 1 \pmod 6$.

Solution by C. D. Olds, Purdue University. Write

$$Q = (x+y)^n - x^n - y^n$$

and

$$p = x^2 + xy + y^2 = (x - \omega y)(x - \omega^2 y),$$

the three cube roots of unity being $1, \omega, \omega^2$. From elementary algebra we know that $1+\omega^n+\omega^{2n}=0$ provided n is not a multiple of 3. Now, p divides Q if Q vanishes when $x=\omega y$ (for then it will vanish again for the conjugate value $x=\omega^2 y$); and p^2 divides Q if $\partial Q/\partial x$ also vanishes when $x=\omega y$.

When $x=\omega y$, we have

$$Q = y^n\{(1+\omega)^n - \omega^n - 1\} = y^n\{(-\omega^2)^n - \omega^n - 1\}$$

and

$$\partial Q/\partial x = n(x+y)^{n-1} - nx^{n-1} = ny^{n-1}\{(1+\omega)^{n-1} - \omega^{n-1}\}$$
$$= ny^{n-1}\{(-\omega^2)^{n-1} - \omega^{n-1}\}.$$

The former vanishes when n is odd and not a multiple of 3; the latter when $n-1$ is even and a multiple of 3, *i.e.*, when $n \equiv 1 \pmod 6$.

This theorem was first proved by Cauchy in 1840 (*Oeuvres*, vol. 4, pp. 499–504). When n is a prime, it can be used to discuss the case $x^p+y^p+z^p=0$ of Fermat's Last Theorem. Numerous references to this problem, together with many applications and generalizations, will be found in Dickson, *History of the Theory of Numbers*, vol. 2, chap. 26.

Also solved by Murray Barbour, F. M. Carpenter and W. W. Gandy (together), Irving Kaplansky, E. D. Schell, Murray Spiegel, E. P. Starke, and the proposer.

The Generalized Water-fetching Puzzle

E 451 [1942, 125–127]. *Proposed by W. E. Buker, Pittsburgh Public Schools*

There are three containers, having capacities of a, b, c quarts, where $a>b>c$ (positive integers). With the largest container full and the others empty, it is desired to divide the liquid into two equal portions, using these containers and no others. For what values of a, b, c is a solution possible?

Partial solution by William Scott, Fort Monroe, Va. In the editorial note to the previous solution it was shown that a sufficient condition for the occurrence of a solution with $a<b+c-2$ is that $\frac{1}{2}a\equiv c \pmod{r}$, where $r\equiv \pm c \pmod{b-c}$ and $0<r<\frac{1}{2}(b-c)-1$. The editor asked for an example to show that this sufficient condition is not also necessary. Here is such an example:

$$a = 18, \qquad b = 17, \qquad c = 4.$$

The sequence of s's is 0, 17, 13, 9, $\cdots$. But $r=4$, and $\frac{1}{2}a$ is not a multiple of 4.

The Three Ladders

E 616 [1944, 231]. *Proposed by J. S. Cromelin, Chicago*

A 60′ ladder and a 77′ ladder rest with their lower ends against a building on the east side of a street, their upper ends against the opposite building on the west wide. A third ladder rests with its lower end against the building on the west side, its upper end against the building on the east side. It crosses the first ladder at a height of 17′, the second at a height of 19′. What is the length of the third ladder, and how wide is the street (to the nearest inch)?

Solution by Murray Barbour, Michigan State College. Let the width of the street be x feet, and let the ladders cross at horizontal distances a and b feet from the west side of the street. Then, by similar right triangles,

$$\frac{x-a}{17}=\frac{x}{\sqrt{60^2-x^2}}, \qquad \frac{x-b}{19}=\frac{x}{\sqrt{77^2-x^2}}, \qquad \frac{a}{17}=\frac{b}{19}.$$

Solving the first two equations for a, b, and substituting in the third, we obtain

$$(60^2-x^2)^{-1/2}-(77^2-x^2)^{-1/2}=2/323.$$

There is no advantage in rationalizing this equation, for the resultant equation would be quartic in x^2. By successive interpolation we soon find $x=36.235\cdots$. Thus, to the nearest inch, the width is 36′ 3″.

The length of the third ladder (in feet) can be expressed as either

$$\frac{x\sqrt{17^2+a^2}}{a} \quad \text{or} \quad \frac{x\sqrt{19^2+b^2}}{b}.$$

Using the above value for x, either formula gives the length 44′ 10″.

Also solved by D. H. Browne, H. N. Carleton, William Douglas, Howard Eves, L. S. Shively, and E. C. Varnum.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4141. *Proposed by H. S. M. Coxeter, University of Toronto*

Prove synthetically that the four lines of 4018 [1943, 125] are (in general) four tangents to a twisted cubic.

4142. *Proposed by G. Pólya, Stanford University*

Find a sequence of real numbers $a_1, a_2, a_3, \cdots$ so that $\sum_1^\infty a_n$ converges, $\sum_1^\infty a_n^3$ diverges, $\sum_1^\infty a_n^5$ converges. More generally, let C be an arbitrarily given (finite or infinite) class of positive integers. There exists a sequence of real numbers $a_1, a_2, \cdots, a_n, \cdots$ adapted to C so that, for $l=1, 2, 3, \cdots$, the series

$$a_1^{2l-1} + a_2^{2l-1} + \cdots + a_n^{2l-1} + \cdots$$

converges or diverges according as l does or does not belong to C.

4143. *Proposed by P. Erdös, Purdue University*

Let $p_1 < p_2 < \cdots < p_n < \cdots$ be the consecutive primes. Prove that

$$\frac{p_n!}{p_n(p_n+1)\cdots(p_{n+1}-1)}$$

is always an integer except when $p_n = 3$.

4144. *Proposed by V. Thébault, San Sebastián, Spain*

For what base is a number of eight digits of the form *ababcdcd* the square of a four digit number *mnmn*, given that *a b c d m n* is a permutation of six consecutive digits?

SOLUTIONS

Affine Geometry

4087 [1943, 391]. *Proposed by Betty Dick and B. M. Stewart, Michigan State College*

Let P be a plane polygon with vertices $A_1, A_2, \cdots, A_n$, and consider $A_{n+k} = A_k$. Determine points $B_1, B_2, \cdots, B_n$ such that B_i is on the side A_iA_{i+1} with $A_iA_{i+1} = k \cdot A_iB_i$, where k is a fixed real number. Let the lines A_iB_{i+1} and

$A_{i+1}B_{i+2}$ intersect in the point C_i, thus determining the polygon Q with the vertices $C_1, C_2, \cdots, C_n$. Let $R(k)$ be the ratio of the area of Q to the area of P, wherein not to restrict the type of polygon we use Klein's definition of area, *Elementary Mathematics from an Advanced Standpoint, Geometry*, p. 10. (1) Show that k and $R(k)$ are invariants under affine transformations. (2) As a corollary to (1) show that for any triangle we have $R(k)=(k-2)^2/(k^2-k+1)$. (3) For any parallelogram we have $R(k)=(k-1)^2/(k^2+1)$. (4) Show that $R(k)$ does not have the same value for all quadrilaterals.

This problem is a generalization of the so-called Problem of Steinhaus which asserts for any triangle $R(3)=1/7$. An early reference, suggestive of this problem, is problem 276 in *Mathesis*.

Solution by Howard Eves, Syracuse University. (1) This part follows from the facts that the ratio of two segments on a line is an absolute invariant, and the area of a polygon is a relative invariant, with respect to the group of affine transformations.

(2) Since any three non-collinear finite points may be carried affinely into any other three non-collinear finite points it follows that any triangle is affinely equivalent to an equilateral triangle. But it is easy to show that, for an equilateral triangle,

$$R(k) = (k-2)^2/(k^2-k+1).$$

By (1) this relation is then true for all triangles.

(3) As in (2), and since parallel lines carry affinely into parallel lines, it follows that any parallelogram is affinely equivalent to a square. It is easy to show that, for a square,

$$R(k) = (k-1)^2/(k^2+1).$$

By (1) this relation is then true for all parallelograms.

(4) To show that $R(k)$ does not have the same value for all quadrilaterals, consider a quadrilateral, S, three of whose vertices are the vertices of an equilateral triangle, and whose fourth vertex, A_4, is near one of the other vertices, say A_3. Then it is readily seen from a figure that, as $A_4 \to A_3$, $R(2) \to 1/6$. But for a parallelogram $R(2)=1/5$.

Solved also by J. H. Butchart and the proposers.

Editorial Note. The solution by the proposers is similar to the above. For (2) the triangle $(0, 0)$, $(k, 0)$, $(0, k)$ is used; for (3) the square with three vertices as above and the fourth (k, k) is used; and for (4) a trapezoid with one base twice the length of the other gives $R(2)=156/775$, whereas for a parallelogram $R(2)=1/5$. For (2) Butchart used vectors for a triangle $A_1A_2A_3$ and eliminated the vector for A_3 from the vector equations for B_2 and B_3 and thus obtained $A_2C_1/C_1B_3=k/(k-1)^2$, $A_2C_2/C_2B_3=k(k-1)$. Additional combinations give the desired result. For (3) the square with side k is used, then Q is a square with side $C_1C_2=k(k-1)/(k^2+1)^{1/2}$ and this gives the desired result for $R(k)$.

Two Special Radical Centers

4088 [1943, 392]. *Proposed by V. Thébault, San Sebastián, Spain*

If three circles $A(\rho)$, $B(\rho)$, $C(\rho)$ with the same radius ρ are described in the triangle ABC, and then the circles with centers A, B, C orthogonal respectively to $C(\rho)$, $A(\rho)$, $B(\rho)$; these three circles have the same radical center M_1 whatever the value of ρ. The same is true of three circles with centers A, B, C orthogonal respectively to $B(\rho)$, $C(\rho)$, $A(\rho)$, the radical center being M_2. If O is the circumcenter of ABC, show also that: (1) The triangles ABC and OM_1M_2 have the same centroid; (2) The straight line M_1M_2 is perpendicular to the straight line through the centroid and the Lemoine point; (3) if M_1' and M_2' are the symmetrics of M_1 and M_2 with respect to O, then M_1M_1' and M_2M_2' are parallel to the Euler line.

Solution by Howard Eves, Syracuse University. We shall use rectangular coordinates with the circumcenter O as origin, and shall vectorize our algebra. Let a, b, c be the sides of the triangle, G the centroid, H the orthocenter, and L the Lemoine point. Denote the coordinates of A, B, C, M_1, M_2, G, H, L by the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{g}$, $\mathbf{h}$, $\mathbf{l}$ respectively.

The equation of the radical axis through M_1 and perpendicular to BC is

$$(\mathbf{x}-\mathbf{w})\cdot(\mathbf{x}-\mathbf{w})-(\mathbf{x}-\mathbf{v})\cdot(\mathbf{x}-\mathbf{v})=(a^2-\rho^2)-(c^2-\rho^2)=a^2-c^2,$$

or, since $\mathbf{w}\cdot\mathbf{w}=\mathbf{v}\cdot\mathbf{v}$

$$2\mathbf{x}\cdot(\mathbf{v}-\mathbf{w})=a^2-c^2. \tag{1}$$

Similarly, the equation of the radical axis through M_1 and perpendicular to CA is

$$2\mathbf{x}\cdot(\mathbf{w}-\mathbf{u})=b^2-a^2. \tag{2}$$

We shall solve this pair of equations by use of the matrix operator ϵ defined as follows. The vector $\mathbf{u}$ has the components u_1, u_2 and may be written as the matrix (u_1, u_2), then

$$\epsilon=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\qquad \mathbf{u}\epsilon=(u_1,u_2)\begin{pmatrix}0&1\\-1&0\end{pmatrix}=(-u_2,u_1)$$

and hence $\mathbf{u}$ and $\mathbf{u}\epsilon$ are perpendicular and equal in length. We have also

$$(\mathbf{u}\epsilon)\cdot(\mathbf{u}\epsilon)=\mathbf{u}\cdot\mathbf{u};\qquad(\mathbf{u}\epsilon)\cdot\mathbf{u}=0;\qquad(\mathbf{u}\epsilon)\cdot\mathbf{v}=\begin{vmatrix}u_1&u_2\\v_1&v_2\end{vmatrix}=|\mathbf{u},\mathbf{v}|.$$

The solution of the pair of equations $\mathbf{x}\cdot\mathbf{u}=r$, $\mathbf{x}\cdot\mathbf{v}=s$ is

$$\mathbf{x}=\frac{s\mathbf{u}-r\mathbf{v}}{|\mathbf{u},\mathbf{v}|}\epsilon$$

which is easily verified by substitution.

Thus the solution of (1) and (2) is

$$\mathbf{m}_1 = \frac{\mathbf{u}(a^2 - c^2) + \mathbf{v}(b^2 - a^2) + \mathbf{w}(c^2 - b^2)}{4\Delta}\epsilon, \tag{3}$$

where Δ is the area of ABC. Similarly, we find

$$\mathbf{m}_2 = \frac{\mathbf{u}(b^2 - a^2) + \mathbf{v}(c^2 - b^2) + \mathbf{w}(a^2 - c^2)}{4\Delta}\epsilon. \tag{4}$$

These results are independent of ρ.

Now it is geometrically evident that H lies on the radical axis of the circles $B(c)$ and $C(b)$. Thus the equation of the altitude on side BC is

$$(\mathbf{x} - \mathbf{w})\cdot(\mathbf{x} - \mathbf{w}) - (\mathbf{x} - \mathbf{v})\cdot(\mathbf{x} - \mathbf{v}) = b^2 - c^2, \quad \text{or} \quad 2\mathbf{x}\cdot(\mathbf{v} - \mathbf{w}) = b^2 - c^2.$$

Similarly the equation of the altitude on side CA is

$$2\mathbf{x}\cdot(\mathbf{w} - \mathbf{u}) = c^2 - a^2.$$

Solving simultaneously we get

$$\mathbf{h} = \frac{\mathbf{u}(b^2 - c^2) + \mathbf{v}(c^2 - a^2) + \mathbf{w}(a^2 - b^2)}{4\Delta}\epsilon.$$

Also we know that

$$\mathbf{l} = \frac{a^2\mathbf{u} + b^2\mathbf{v} + c^2\mathbf{w}}{a^2 + b^2 + c^2}, \qquad \mathbf{g} = \frac{\mathbf{u} + \mathbf{v} + \mathbf{w}}{3}.$$

From (3), (4), (5) we see that $\mathbf{m}_1+\mathbf{m}_2=\mathbf{h}$. But, since O, G, H are collinear and $OH=3OG$, it follows that $\mathbf{h}=3\mathbf{g}$. Hence $\mathbf{m}_1+\mathbf{m}_2=3\mathbf{g}$ and G is the centroid of M_1M_2O.

We also see that the vectors $\mathbf{m}_1-\mathbf{m}_2$ and $(\mathbf{l}-\mathbf{g})\epsilon$ are proportional, whence it follows that M_1M_2 is perpendicular to LG. Finally, since M_1M_2' and M_2M_1' are parallel to GO, it follows that each is parallel to the Euler line. Incidentally the nine point center is the midpoint of M_1M_2, and M_1HM_2O is a parallelogram.

Solved also by the proposer.

Editorial Note. The proposer used the theorem on the difference of the powers of the vertex A with respect to the circles with centers at B and C orthogonal to $A(\rho)$ and $B(\rho)$ respectively, and considered the two cases of radical axes. For the remaining part the triangle $A_1B_1C_1$ was introduced formed by the perpendiculars to BC at C, to CA at A, and to AB at B, so that O is the Lemoine point for $A_1B_1C_1$. The barycentric coordinates of M_1, M_2, and their associates with respect to $A_1B_1C_1$ and $A_2B_2C_2$, the symmetric of $A_1B_1C_1$ with respect to O, were used to complete the proof.

In the solution above the vector equations (1) and (2) suggest the following form for their solution

$$\mathbf{m}_i = A_i(\mathbf{v} + \mathbf{w}) + B_i(\mathbf{w} + \mathbf{u}), \qquad i = 1, 2,$$

where the scalars A_i, B_i are to be determined by the pair of equations which $\mathbf{m}_i$ satisfy. An examination of the figure enables us to write down immediately six scalar products, the remaining four of which are cyclic permutations of

$$(\mathbf{w}+\mathbf{u})\cdot(\mathbf{u}-\mathbf{v}) = (\mathbf{w}+\mathbf{u})\cdot(\mathbf{w}-\mathbf{v}) = ac\cos\beta,$$

where α, β, γ are the angles opposite the sides a, b, c. We then find that

$$2A_1bc\cos\alpha = b^2 - a^2, \qquad 2B_1ca\cos\beta = c^2 - a^2$$
$$2A_2bc\cos\alpha = c^2 - b^2, \qquad 2B_2ca\cos\beta = a^2 - b^2.$$

We have also $(\mathbf{h}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{w}) = (\mathbf{h}-\mathbf{v})\cdot(\mathbf{w}-\mathbf{u}) = 0$, and these give after trigonometric reductions

$$2\mathbf{h}\cdot(\mathbf{v}-\mathbf{w}) = b^2 - c^2, \qquad 2\mathbf{h}\cdot(\mathbf{w}-\mathbf{u}) = c^2 - a^2.$$

Since $\mathbf{m}_1+\mathbf{m}_2$ satisfy the same two equations, we have $\mathbf{h} = \mathbf{m}_1+\mathbf{m}_2$.

Finally we have after reductions

$$2abc\cos\alpha\cos\beta(\mathbf{m}_1-\mathbf{m}_2) = a\cos\beta(2b^2-c^2-a^2)(\mathbf{v}+\mathbf{w}) - b\cos\alpha(2a^2-b^2-c^2)(\mathbf{u}+\mathbf{w}),$$
$$3(a^2+b^2+c^2)(1-\mathbf{g}) = (2b^2-c^2-a^2)(\mathbf{v}-\mathbf{w}) + (2a^2-b^2-c^2)(\mathbf{u}-\mathbf{w});$$

and it is easily verified that they give $(\mathbf{m}_1-\mathbf{m}_2)\cdot(1-\mathbf{g}) = 0$. It should be noted in the above that, if the triangle has a right angle, we may assume that it is γ; also the case of the equilateral triangle is exceptional. The three vectors satisfy the relation

$$a\cos\alpha\mathbf{u} + b\cos\beta\mathbf{v} + c\cos\gamma\mathbf{w} = 0.$$

We may also use as an operator the unit vector $\mathbf{k}$ normal to the horizontal plane of the figure and directed vertically upward. The vector $\mathbf{k}\times\mathbf{u}$ is the vector $\mathbf{u}$ turned through the positive right angle without altering its length; $\mathbf{u}, \mathbf{k}\times\mathbf{u}, \mathbf{k}$ forming a rectangular system in positive order of rotation. We then have $[\mathbf{k}\times(\mathbf{v}-\mathbf{w})]\cdot(\mathbf{w}-\mathbf{u}) = [\mathbf{k}\times(\mathbf{u}-\mathbf{w})]\cdot(\mathbf{v}-\mathbf{w}) = 2\Delta$; and the solution of the system (1), (2) is

$$4\Delta\mathbf{m}_1 = \mathbf{k}\times[(b^2-a^2)(\mathbf{v}-\mathbf{w}) + (c^2-a^2)(\mathbf{w}-\mathbf{u})]$$

and Eves' proof may be applied with this operator.

The Monge Point

4090 [1943, 456]. *Proposed by N. A. Court, University of Oklahoma*

The polar plane, with respect to the "quasipolar" sphere of a tetrahedron (T), of a point on the circumsphere of (T) trisects the segment joining the Monge point of (T) to the diametric opposite of the given point on the circumsphere.

Note. The "quasipolar" sphere (Q) of a tetrahedron (T) has for its center the Monge point M of (T) and for the square of its radius one third of the power of M for the circumsphere (O) of (T). See *Bull. Am. Math. So.* vol. 48, 1942, p. 583.

Solution by the Proposer. Let L be the given point on the sphere (O), and P the second point of intersection of (O) with the line LM. The traces L' of the polar plane λ of L for the quasi-polar sphere (Q) on LMP is the inverse of L for (Q), hence $ML \cdot ML'$ is equal to the square of the radius of (Q). On the other hand $ML \cdot MP$ is the power of M for the sphere (O), hence, taking into account the definition of (Q), we have

$$ML \cdot MP = 3ML \cdot ML', \quad \text{or} \quad MP = 3ML'.$$

Let L'' be the diametric opposite of L on (O). The plane λ is perpendicular to the line LML', hence λ cuts the plane LPL'' along a line perpendicular to LML' at L' and therefore parallel to the line PL''. Hence the proposition.

Remark. This is an extension to the general tetrahedron of a known property of the orthocentric tetrahedron. See the proposer's *Modern Pure Solid Geometry*, pp. 266, 267, art. 814.

Editorial Note. This theorem can be extended to Euclidean space of n dimensions, $n \geqq 2$, where we suppose that the Monge point M of the simplex S is not on its circumsphere (C). Taking M as the origin of vectors, we have the following equations

$$(M): \quad \mathbf{x}^2 - m = 0; \qquad (C): \quad \mathbf{x}^2 - 2\mathbf{c} \cdot \mathbf{x} + nm = 0;$$

where $\mathbf{c}$ is the vector of C, the circumcenter of S, $m \neq 0$ by the above assumption, and (M) is the "quasi polar" sphere of S. The above equations are given in the solution of 4049 [1943, 578]. Let P_1 on (C) have the vector $\mathbf{x}_1$, then P_1', its diametric opposite on (C), has the vector $2\mathbf{c} - \mathbf{x}_1$, and the polar plane of P_1 with respect to (M) has the equation $\mathbf{x}_1 \cdot \mathbf{x} - m = 0$. Let this polar plane and the straight line MP_1' meet in Q; then Q has the vector $t(2\mathbf{c} - x_1)$, where $t(2\mathbf{c} \cdot \mathbf{x}_1 - \mathbf{x}_1^2) - m = 0$, and from the equation of (C) we get $(tn-1)m = 0$. From this follows that $MQ = MP_1'/n$, which is the desired result. It is obvious from the equation of (C) that the power of M with respect to (C) is nm.

Three Related Series

4091 [1943, 457]. *Proposed by Morgan Ward, California Institute of Technology*

Given the three series

$$z - \frac{z^5}{2\cdot4\cdot5} + \frac{z^9}{2\cdot4\cdot6\cdot8\cdot9} - \frac{z^{13}}{2\cdot4\cdot6\cdot8\cdot10\cdot12\cdot13} + \cdots,$$

$$\frac{z^3}{2\cdot3} - \frac{z^7}{2\cdot4\cdot6\cdot7} + \frac{z^{11}}{2\cdot4\cdot6\cdot8\cdot10\cdot11} - \frac{z^{15}}{2\cdot4\cdot6\cdot8\cdot10\cdot12\cdot14\cdot15} + \cdots,$$

$$\frac{z^2}{1\cdot2} - \frac{z^6}{1\cdot3\cdot5\cdot6} + \frac{z^{10}}{1\cdot3\cdot5\cdot7\cdot9\cdot10} - \frac{z^{14}}{1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot13\cdot14} + \cdots,$$

prove that the sum of the squares of the first two series is double the third series.

Solution by B. P. Gill, City College, N. Y. Denote the three series in the order given by f, g, h. Then term by term differentiation, which is clearly permissible, yields, if accents denote derivatives with respect to z, (1) $f''=-zg'$, $g''=zf'$. From these follow (2) $f'f''+g'g''=0$, whence $f'^2+g'^2=C$. Putting $z=0$ gives $C=1$, so that (3) $f'^2+g'^2=1$. Equations (2) and (3) also follow from the fact, which we shall not need, that $f'=\cos(z^2/2)$, $g'=\sin(z^2/2)$. Now differentiation of (1) and substitution show that f and g are both solutions of the differential equation $L[w]=zw'''-w''+z^2w'=0$, that is, $L[f]=L[g]=0$.

Applying a similar argument to the two functions h and

$$k \equiv 1-\frac{z^4}{1\cdot 3\cdot 4}+\frac{z^8}{1\cdot 3\cdot 5\cdot 7\cdot 8}-\frac{z^{12}}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 12}+\cdots,$$

we get $h''=1+zk'$, $k''=-zh'$, $L[h]=-1$. But if we write $y=f^2+g^2$, straightforward computation using (2) and (3) gives $L[y]=2(fL[f]+gL[g]-1)=-2$. Hence $2h$ and y are both solutions of $L[w]=-2$, while also $y(0)=2h(0)=0$, $y'(0)=2h'(0)=0$, $y''(0)=2h''(0)=2$. Hence, because of the uniqueness of the solution of the differential equation under these last initial conditions, $y=2h$, which is what was to be proved.

Solved also by A. B. Farnell, F. A. Ficken, Elijah Swift, H. E. Vaughan and the proposer.

Reciprocal Simplexes

4079 [1943, 264]. *Proposed by S. Beatty, Univ. of Toronto*

If two simplexes, in projective n-space, are reciprocal with respect to a quadric, so that each vertex of one is the pole of a bounding hyperplane of the other, then the $n+1$ lines joining corresponding vertices are associated,* in the sense that every $(n-2)$-space which meets n of them meets the remaining one also. Reciprocally, the $n+1$ of the $(n-2)$-spaces of intersection of corresponding hyperplanes are such that every line which meets n of them meets the remaining one also.

Solution by J. A. Todd, Cambridge University, and H. S. M. Coxeter, University of Toronto. Let one simplex be the simplex of reference $X_1X_2\cdots X_{n+1}$, and $\sum\sum A_{i,j}X_iX_j=0$ be the tangential equation of the quadric, so that the other simplex is $Y_1Y_2\cdots Y_{n+1}$ where (*e.g.*) Y_1, being the pole of $x_1=0$, is $(A_{1,1}, A_{1,2}, \cdots, A_{1,n+1})$. A variable point of the line X_1Y_1 is $(\lambda_1, A_{1,2}, \cdots, A_{1,n+1})$, where λ_1 is a parameter. The condition for n such points, one on each of the lines X_1Y_1, X_2Y_2, $\cdots$, X_nY_n, to lie in an $(n-2)$-space, is the same as the condition for the first $n-1$ points and the whole line X_nY_n to lie in a hyperplane, namely

* For example, when $n=3$, the four lines are generators of a regulus. The case where $n=2$ is Hesse's theorem.

$$\begin{vmatrix} \lambda_1 & A_{1,2} & \cdots & A_{1,n-1} & A_{1,n} & A_{1,n+1} \\ A_{2,1} & \lambda_2 & \cdots & A_{2,n-1} & A_{2,n} & A_{2,n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n-1,1} & A_{n-1,2} & \cdots & \lambda_{n-1} & A_{n-1,n} & A_{n-1,n+1} \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n-1} & A_{n,n} & A_{n,n+1} \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{vmatrix} = 0, \quad \text{or}$$

$$\begin{vmatrix} \lambda_1 & A_{1,2} & \cdots & A_{1,n-1} & A_{1,n+1} \\ A_{2,1} & \lambda_2 & \cdots & A_{2,n-1} & A_{2,n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n-1,1} & A_{n-1,2} & \cdots & \lambda_{n-1} & A_{n-1,n+1} \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n-1} & A_{n,n+1} \end{vmatrix} = 0.$$

This is a relation between the $n-1$ parameters $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}$ (any $n-2$ of which may be chosen arbitrarily). But, since $A_{i,j}=A_{j,i}$, the relation is symmetrical between the suffixes n and $n+1$. Thus every $(n-2)$-space which meets the lines $X_1Y_1, X_2Y_2, \cdots, X_nY_n$ meets $X_{n+1}Y_{n+1}$ too.

Editorial Note. After the appearance of the problem in print, Coxeter reported that it had been pointed out to him that the three dimensional case is treated in a pure geometrical manner in H. F. Baker's *Principles of Geometry*, vol. 3, p. 41, Ex. 7. He stated also that Baker's proof is readily extensible to higher space by induction. A later communication reported that an analytical treatment of the general case, different from the above solution, is given by H. F. Baker, *Polarities for the nodes of a Segre cubic primal in space of four dimensions*, Proc. Camb. Phil. Soc. 32, 1936, p. 510.

Excenter in Hyperbolic Geometry

4082 [1943, 329]. *Proposed by H. S. M. Coxeter, University of Toronto*

In hyperbolic geometry, obtain the condition

$$\cos \tfrac{1}{2}A < \sin \tfrac{1}{2}B + \sin \tfrac{1}{2}C$$

for the triangle ABC to have a proper excircle beyond BC.

Solution by the Proposer. Let E be the excenter, if there is one, so that BE and CE are the external bisectors of the angles B and C. From the triangle ABC, with $BC=a$, we have

$$\sin B \sin C \cosh a = \cos B \cos C + \cos A.$$

Similarly, from the triangle EBC,

$$\cos \tfrac{1}{2}B \cos \tfrac{1}{2}C \cosh a = \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C + \cos E.$$

Eliminating a, we have

$$4 \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C(\sin \tfrac{1}{2}B \sin \tfrac{1}{2}C + \cos E) = \cos B \cos C + \cos A$$
$$= 2 \cos^2 \tfrac{1}{2}A - 2 \sin^2 \tfrac{1}{2}B - 2 \sin^2 \tfrac{1}{2}C + 4 \sin^2 \tfrac{1}{2}B \sin^2 \tfrac{1}{2}C,$$

whence

$$\cos^2 \tfrac{1}{2}A = \sin^2 \tfrac{1}{2}B + \sin^2 \tfrac{1}{2}C + 2 \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \cos E.$$

The condition for E to be an ordinary (or accessible) point is $\cos E < 1$. This is a much simpler form of the condition than that given on p. 239 of my *Non-Euclidean Geometry*, 1942.

Tangents to Kiepert's Hyperbola

4089. [1943, 456]. *Proposed by J. H. Butchart, Grinnell College*

The tangent to Kiepert's hyperbola at a vertex of the triangle is the harmonic conjugate of the symmedian with respect to the altitude and median from that vertex. It meets the corresponding side of the medial triangle at a point on the tangent to the hyperbola at the centroid.

Editorial Note. The proposer gave the following remark:

To establish these, I use the similar isosceles triangles constructed on the sides of the given triangle together with the definition of a conic by projectively related pencils of rays. The second part is a consequence of the properties of the inscribed quadrangle.

We shall add some details which may aid in following the proof. In the solu- of problems 3882, 3883 [1940, 401–404] it is stated on p. 404, that if on the sides of triangle ABC directly similar isosceles triangles P_aBC, P_bCA, P_cAB are constructed, all directed interiorly, or all exteriorly, the straight lines AP_a, BP_b, CP_c are concurrent in a point P; and, as the base angle of the similar triangles varies, P describes a conic κ through A, B, C, G, H, the last two points being the centroid and orthocenter of ABC. Indications of the proof are given by means of projective pencils of rays with centers at two vertices. The conic κ is Kiepert's hyperbola, the isogonal conjugate of the circumdiameter of ABC through the symmedian point K. In that proof the case where the isosceles triangles are interiorly directed with the base angle A, the angle opposite BC, is considered, and this gives the point P at the vertex A. Denote by K_a' the point P_a for this base angle, then $K_a'A$ meets κ in only one point A, whereas for a base angle slightly different from A, the line P_aA meets κ in A and also in a second point P. Hence $K_a'A$ is the tangent at A to κ. Now consider the isosceles triangle K_aCB symmetric to $K_a'BC$ with respect to A_0 the midpoint of BC; then the pencil of rays AK_a', AA_0, AK_a, AH is harmonic, since AH and $K_a'K_a$ are parallel. It is known that AK_a passes through K; this proves the first part.

The second theorem is true for any conic through A, B, C, G; this conic must be a hyperbola since G is inside triangle ABC. Consider the degenerate hexagon $AA\ B\ GG\ C$, where AA, GG denote the tangents t_a, t_g to the conic at A, G. The three points (t_a, t_g), $(AB, GC) = C_0$, $(BG, CA) = B_0$ are collinear, and this proves

a more general theorem: A necessary and sufficient condition for a conic through A, B, C to be a rectangular hyperbola is that it passes through H. For such a conic we prove in the same way that the tangents at A and H meet on B_AC_A, the corresponding side of the pedal triangle for H with respect to ABC.

Ruled Quadrics

4094 [1943, 516]. *Proposed by N. A. Court, University of Oklahoma.* A. If a line moves so that the segment intercepted on it by two fixed skew lines subtends a right angle at a fixed point in space, the line generates a ruled quadric (This proposition is due to E. Bobillier, 1797–1832. It was proved analytically in *Nouvelles Annales de Mathématiques*, 1862, pp. 318, 320 and 379, 380).

B. The segment intercepted by two elements of the same system of a ruled quadric on a variable element of the complementary system subtends a right angle at each of two fixed points in space.

Solution by James Jenkins, Student, University of Toronto. A. Let a, b be the two fixed skew lines and P the fixed point. Let AB be the moving line, with A on a and B on b. The flat pencil of lines PA is projectively related to an axial pencil of perpendicular planes through P (with axis perpendicular to Pa). If the planes Pa, Pb are not perpendicular, the latter will meet this axial pencil in a related flat pencil of lines PB; so $A \overline{\wedge} B$, and AB traces out a regulus. If the planes Pa, Pb are perpendicular, we have instead two flat pencils of lines AB, one joining a particular point on a to every point on b, and the other vice versa.

B. Let a, b be two lines of a regulus. Take any three lines of the associated regulus (or complementary system), and using as diameters the segments cut off on these by a and b, describe three spheres. In general, two of these spheres will meet in a circle which will meet the third in two points. Each of the three segments then subtends a right angle at either of these points. But (by A) all lines giving segments which subtend right angles at these two points belong to a regulus. Since three lines suffice to determine a regulus, these are just the lines of the given regulus. Of course, the two common points of the three spheres may be real, coincident, or conjugate imaginary.

The words "in general" here mean "provided the midpoints of the three segments are not collinear," or "provided the ruled quadric is a hyperboloid." If, on the other hand, the ruled quadric is a paraboloid, then the three spheres have collinear centers and consequently have either no common points or a common circle. In the former case there is no point from which the segments all subtend right angles; in the latter there are infinitely many. Since the points at infinity on a and b form one such segment, the latter case arises only when a and b are "perpendicular."

Solved also by the proposer.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Professor Solomon Lefschetz of Princeton University has been elected an honorary member of the Mexican Mathematical Society.

Associate Professor Reinhold Baer of the University of Illinois has been promoted to a professorship.

Dr. T. A. Bancroft of Iowa State College has been promoted to an assistant professorship.

Dr. F. C. Biesele of the University of Utah has been promoted to an assistant professorship.

Dr. E. E. Blanche of Curtiss-Wright Corporation has been appointed principal statistician and chief of operations for the Foreign Economic Administration in Washington, D. C.

Associate Professor N. R. Bryan of the University of Maine has been promoted to a professorship.

Dr. R. E. Byrne has been appointed to an assistant professorship at the University of California at Los Angeles.

Dr. R. H. Cole of the University of Western Ontario has been promoted to an assistant professorship.

Assistant Professor Mary E. Decherd of the University of Texas has retired.

Assistant Professor H. L. Dorwart of Washington and Jefferson College has been promoted to an associate professorship.

Professor W. S. Erickson of the State Teachers College at Minot, North Dakota, has been appointed to a fellowship in mechanics at Brown University.

Associate Professor W. M. Ewing of Lehigh University has been appointed to an associate professorship at Columbia University.

Dean T. M. Focke of the Case School of Applied Science has been retired with the title emeritus.

Dr. Hans Fried of Swarthmore College has been appointed lecturer.

E. S. Grable of the University of Richmond has been promoted to an assistant professorship.

Dr. Mary C. Graustein has been appointed to an assistant professorship at Tufts College.

Associate Professor E. A. Hedberg of Baylor University, Waco, Texas, has been appointed visiting associate professor in electrical communications at Massachusetts Institute of Technology.

Dr. Olaf Helmer of the College of the City of New York has been appointed lecturer at the New School for Social Research, New York City.

Assistant Professor J. T. Hurt of the Agricultural and Mechanical College of Texas has been promoted to an associate professorship.

Associate Professor A. E. Johns of McMaster University has been promoted to a professorship.

Dr. R. E. Johnson has been appointed to an assistant professorship at Mount Holyoke College.

Assistant Professor Roberta F. Johnson of Wilson College, Chambersburg, Pennsylvania, has been promoted to an associate professorship.

Assistant Professor H. L. Krall of Pennsylvania State College has been promoted to an associate professorship.

Dr. P. E. Lewis of Oklahoma Agricultural and Mechanical College has been promoted to an assistant professorship.

Dr. R. R. R. Luckey of Houghton College has been promoted to an associate professorship.

Professor E. L. Mackie of the University of North Carolina has been appointed Dean of Men.

Dr. Leonard McFadden of Virginia Polytechnic Institute has been promoted to an associate professorship.

Dr. E. J. Mickle of Ohio State University has been promoted to an assistant professorship.

Assistant Professor R. H. Moorman of Tennessee Polytechnic Institute has been granted leave of absence to serve as acting associate professor at the College of Charleston, South Carolina.

Dr. C. G. A. Nordling of the University of Connecticut has been promoted to an assistant professorship.

Miss Elizabeth A. Oliphant of Texas College of Arts and Industries, Kingsville, Texas, has been promoted to an assistant professorship.

Assistant Professor T. S. Peterson of the University of Oregon has been promoted to an associate professorship.

Reverend J. H. Raymond of St. Martin's College, Lacey, Washington, has been promoted to an assistant professorship.

Dr. I. F. Ritter of New York University has been promoted to an assistant professorship.

Associate Professor J. S. Rosen of Eastern New Mexico College has been appointed to an assistant professorship at the University of Kansas City.

Dr. E. H. Rothe has been appointed to an assistant professorship at the University of Michigan.

Assistant Professor P. A. Samuelson of Massachusetts Institute of Technology has been promoted to an associate professorship.

Assistant Professor Edith R. Schneckenburger of Michigan State Normal College, Ypsilanti, Michigan, has been promoted to an associate professorship.

Dr. Abraham Schwartz of Pennsylvania State College has been promoted to an assistant professorship.

R. E. Smith of Allegheny College has been granted a year's leave of absence to teach engineering and mathematics at Brown University.

Associate Professor D. V. Steed of the University of Southern California has been promoted to a professorship.

Assistant Professor A. E. Taylor of the University of California at Los Angeles has been promoted to an associate professorship.

Professor E. H. Taylor of Eastern Illinois State Teachers College has retired.

Assistant Professor Abraham Wald of Columbia University has been promoted to an associate professorship.

Assistant Professor S. E. Warschawski of Washington University has been promoted to an associate professorship.

Associate Professor Antoni Zygmund of Mount Holyoke College has been promoted to a professorship.

The following appointments to instructorships have been announced:
Connecticut College: Dr. Josephine M. Mitchell
Michigan State College: Dr. Margaret Mauch
Smith College: Aida Kalish
University of California: Dr. S. A. Schaaf
University of Chicago: Dr. R. M. Martin
University of Michigan: Dr. A. A. Grau

Professor G. D. Birkhoff of Harvard University died November 12, 1944. He was a charter member of the Mathematical Association.

Professor J. A. Shohat of the University of Pennsylvania died October 8, 1944.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

GENERAL EDUCATION FOR THE ARMED FORCES

Ten basic objectives and twelve specific courses for general education for members of the armed forces are outlined in a report which a committee of the American Council on Education has recently completed at the request of the United States Armed Forces Institute. Dean T. R. McConnell of the University of Minnesota was chairman of the committee. The suggested instructional materials, planned primarily for the period following the end of hostilities, are now being prepared by the Institute for use in correspondence study and group instruction. According to Dr. George F. Zook, president of the Council, "The proposed program should serve as an effective bridge between military activities and the return of men and women to civilian life." The courses have been developed for men and women who would normally register for work at the upper senior high school or junior college level. However, it is assumed that persons who have served in the armed forces will be more mature than the usual high school or junior college student, so the courses have been prepared for an adult point of view.

The report has recently been made available for civilian use under the title, "A Design for General Education." Dr. Zook said, "The committee had not proceeded far with its work before it discovered that civilian educational institutions—particularly secondary schools, junior colleges, and colleges—would be as much interested in this report as the armed forces. These institutions recognize that the postwar educational programs for service personnel will probably demand new curricular patterns. Many schools and colleges are already studying their programs to get ready for this new responsibility. In addition, teachers and administrators throughout the country are reconsidering their provisions for general education." So it is hoped the study will assist institutions in serving returning service personnel who have enrolled in courses similar to those described.

The committee defines a general education as "the type of education which the majority of our people must have if they are to be good citizens, parents, and workers." Ten fundamental objectives are then formulated for such a program. Objective VI, of interest to mathematicians, specifies that a "general education should lead the individual as a citizen in a free society . . . to use scientific methods in the solution of his problems, ..." An explanatory footnote emphasizes that mathematical skills involved in this objective are usually obtained before the student reaches the level of general education considered in the outline; if such skills are not present, it is suggested that the student review courses offered

at the elementary or high school level. "Computational skill beyond the level of high school algebra may be developed as part of a student's program of specialization but is not included as an essential element in his general education."

The courses proposed by the report are as follows: personal and community health; oral and written communications; problems of social adjustment; marriage and family adjustment; development of American thought and institutions; problems of American life; America in international affairs; science—biological and physical; literature—American life and ideals in literature readings; form and function of art in society; music in relation to human experience; philosophy and religion—the meaning and value of life; and vocational orientation. Extensive bibliographies are provided for each course, except for the one on science.

Some mathematics is included as one of the "units" in the outline of the course in physical science. The following syllabus of the unit attempts to combine the general discussion of the unit and its topical outline, as they appear in the report.

1. Numbers. The evolution of the real number system, and some mention of imaginary numbers.
2. Algebra. Importance of "generalized numbers"; the function concept; some review of algebraic operations. "The convenience of algebra" is to be emphasized.
3. Geometry. The abstract nature of geometry; the use of symbolism in making proofs.
4. Postulational Structure of Mathematics. Deductive reasoning as applied in mathematics; Euclidean and non-Euclidean geometries; noncommutative algebras; "mathematical truth."
5. Trigonometry. Definitions of trigonometric functions and illustrations of their use; graphs.
6. Analytic Geometry. "Graphing of a few equations."
7. Differential Calculus. The derivative presented as a time-rate.
8. Integral Calculus. "Qualitative description of the methods in geometric language only."

EDUCATIONAL PROGRAM FOR THE ARMY OF OCCUPATION

The following bulletin was released by the War Department on September 29, 1944.

"Soldiers serving in the Army of Occupation or awaiting shipment home after the defeat of Germany will have an opportunity to further their education or receive practical training to prepare for civilian jobs under an extensive program provided by the Army.

"To the extent that the military mission of a unit will allow, parts of the duty day that are now devoted to strictly military training will be utilized for instruction in academic or vocational subjects, or supervised athletics and recreation. Troops will have free choice as to which phases of the program they follow,

but will be encouraged to enroll in activities that have a bearing on their individual postwar plans and ambitions.

"The program will be applicable in all inactive theaters of operation when the military situation in each permits, but will be especially useful during the anticipated period between the defeat of Germany and the fall of Japan, when large numbers of troops will remain in Europe for a considerable length of time. While many will be sent to the Pacific, many others will stay for occupation duties, and some will be sent home.

"Since troops have been sent abroad at an ever-increasing rate for almost three years, the return of any of the men will require some time, and will depend upon the availability of shipping. A substantial part of the inevitable waiting period can be used profitably in one or more of the educational activities.

"Academic curricula will range in level from the sixth grade through second year college, and will include courses in the liberal arts and in the scientific and pre-professional fields. Many of the courses are designed to aid soldiers who plan to continue their educations after leaving the Army. Facilities of inactive educational institutions may be used for study centers, and opportunities may be given to attend courses at foreign colleges or universities. The nature of the subjects taught at each school will be determined by the preferences of the soldiers.

"Soldiers whose ability to read and write is below the fifth grade standard will be encouraged to enroll in special classes.

"Officers and enlisted men who meet the qualifications will be designated—without regard to rank—to teach the courses. Instructors' outlines and textbooks have been prepared by leading educators, and all teachers will undergo a brief training period.

"In addition to academic subjects, courses in mechanical and technical subjects will be given, and practical training in trades and vocations will be provided for men who plan to go directly into civilian employment after discharge. Specialist personnel will teach the classes, and equipment of the technical services such as Signal Corps, Quartermaster Corps and Ordnance Department will be used for practical work.

"No one will be delayed in returning to the United States by participation in the program. When a soldier receives shipping orders he will pack up and leave immediately, regardless of the stage of any course he might be taking.

"The material for each course is divided into units of work, each unit requiring 20 hours of classroom study. Insofar as possible, each unit is made independent of subsequent units, so that if a student stops in the middle of a course to go home, he still will have benefited from the instruction.

"Schools were operated for servicemen overseas after the World War, when more than 9,000 soldiers attended an Army university at Beaune, France, and several thousand more took courses at British and French universities. With a far larger number of Army personnel overseas during this war, the scope of the present program will be infinitely greater.

"During the present war, more than 800,000 servicemen, both in the United States and overseas, have taken correspondence and self-teaching courses in their off-duty hours from the United States Armed Forces Institute, an agency sponsored jointly by the Army and Navy. Textbooks and educational advice is provided, as well as aid in obtaining academic credit in schools and colleges. When hostilities cease in each of the war theaters, the educational activities of the Institute will be carried on during part of the duty-time of the servicemen.

"The entire program will, of course, be subject to considerations of military necessity. Theater commanders will decide which units under their command may substitute educational activities for portions of their military training programs."

REGISTRATION IN USAFI COURSES

Bulletin No. 72 of "Higher Education and National Defense," issued by the American Council on Education, reveals the trend of registrations in the correspondence courses sponsored by the United States Armed Forces Institute. During the months of April, May, and June, 1944, a total of 22,463 men in the Army enrolled for correspondence courses. Of this number, nearly one-fourth (5,118) registered for work in mathematics; moreover, 1,722 men wanted courses more advanced than elementary arithmetic and algebra. Other courses which proved to be popular, with the enrollment in each, were bookkeeping and accounting (3,167), engines and boilers (1,593), auto mechanics (1,505), radio and communication (1,497), and English grammar (1,112).

For the same period of time, a total of 11,444 men from the Navy signed for correspondence courses. Of this number, 3,368 men registered for work in mathematics; moreover, 985 wanted work more advanced than elementary arithmetic and algebra. Other courses popular with men from the Navy were bookkeeping and accounting (985), engines and boilers (882), refrigeration and air conditioning (712), English grammar (712), and American history (701).

During the same three months, a total of 10,740 men from both the Army and the Navy enrolled through the Institute for correspondence courses sponsored by colleges and universities. The four most popular classifications of courses were social studies (2,027), English (2,024), mathematics (1,951), and business (1,435).

The American Council on Education warns that caution should be used in the interpretation of this data. For one thing, a basic motive of men in their selection of USAFI correspondence courses is the "improvement of the individual's military effectiveness and to procure advance in rating."

THE FUTURE OF THE NAVY V-12 PROGRAM

An announcement by the Navy Department on October 15, 1944, revealed that it is not planned to enroll any new trainees in any of the Navy college training programs for the term beginning March 1, 1945. The announcement emphasized, however, that students already in training "will continue the train-

ing as scheduled." The decision to eliminate new enrollments for that period was based upon revised estimates of future officer requirements.

At the time of the announcement, approximately 69,000 men were enrolled in the V-12 Program. This figure, however, is expected to drop to about 50,000 on November 1, when many of the men first registered in the Program will have completed the prescribed number of terms. Based upon an estimate of the number of men in the various classes of the Program, it appears that the total enrollment may drop to 30,000 men on March 1.

It should be distinctly understood that the Navy has given no indication at the present time that it is planned to discontinue the V-12 Program at an early date. The future of the Program will undoubtedly depend upon military exigencies.

ENGINEERING EDUCATION AFTER THE WAR

The following is an excerpt from the report of the Committee on Engineering Education after the War, sponsored by the Society for the Promotion of Engineering Education, as summarized by Dean H. P. Hammond, Chairman, before the annual meeting of the Society on June 24, 1944.

"The committee calls attention to two recent trends in engineering and industry for which provision should be made in educational programs. One of these trends is the rapid increase in the application of the engineering method to the management and operation of industry as is evidenced by the phenomenal success of graduates in these activities. The other is the greatly increased need for engineers equipped to practice at high scientific and creative level. The remark was made recently by an engineer having unusual knowledge of technical developments during the war, that if engineers are not to be equipped to work on new developments at high creative level, physicists are likely to take over the designing of the devices which will use the scientific principles they are discovering—as indeed they are already doing in some instances.

"In order to provide for the satisfaction of the needs incident to these trends, the committee suggests, for consideration, a plan of curricular differentiation in the fourth year, through which three options would be offered within each major professional curriculum:

(1) Continuation of the present type of 4-year program essentially as a terminal curriculum, but with modifications advocated by the committee, for a majority of students.

(2) An alternative fourth year emphasizing subjects dealing with the management of construction and production enterprises.

(3) A fourth year intended to prepare for additional years of advanced study by strengthening the student's command and extending his knowledge of basic sciences and mathematics, and by introducing him to the methods of advanced study. This fourth year and the year or years of graduate study to follow would be planned as a unit rather than as two stages marked by the usual differences of undergraduate and postgraduate programs."

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE ANNUAL MEETING OF THE METROPOLITAN NEW YORK SECTION

The third annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at New York University, Washington Square, New York City, on Saturday, April 22, 1944. Professor R. M. Foster, Chairman of the Section, presided at the morning session and at the business meeting. Mr. Max Peters, Vice-Chairman of the Section, presided at the afternoon session.

The attendance was one hundred and two, including the following forty-seven members of the Association: Claire F. Adler, R. G. Archibald, I. L. Battin, Brother Bernard Alfred (Welch), Frank Boehm, C. B. Boyer, A. B. Brown, Jewell Hughes Bushey, Ruth T. Coleman, H. R. Cooley, T. F. Cope, W. H. H. Cowles, D. R. Davis, J. E. Eaton, W. H. Fagerstrom, J. M. Feld, Edward Fleisher, R. M. Foster, P. H. Graham, Marion C. Gray, Mary W. Gray, Harriet M. Griffin, C. C. Grove, C. E. Heilman, J. H. Hlavaty, Joseph Jablonower, Nathan Lazar, C. H. Lehmann, H. F. Mac Neish, May Hickey Maria, Joseph Milkman, F. H. Miller, M. A. Nordgaard, Max Peters, Mina S. Rees, Moses Richardson, S. G. Roth, Charles Salkind, A. A. Schwartz, James Singer, E. R. Stabler, H. E. Wahlert, Israel Wallach, Alan Wayne, John Williamson, Jack Wolfe, R. C. Yates.

At the beginning of the morning session Dean Charles M. McConn of the Washington Square College of New York University welcomed the Section to New York University. At the business meeting the following officers were elected for the coming year: Chairman, Jewell Hughes Bushey, Hunter College; Vice-Chairman, Nathan Lazar, Midwood High School; Secretary, H. E. Wahlert, New York University; Treasurer, F. H. Miller, Cooper Union.

The following program was presented:

1. *Elementary mathematical theory of exterior ballistics*, by Professor H. F. Mac Neish, Brooklyn College.

Three aspects of the subject were considered. First, the parabolic trajectory in the ideal case in which there is no resistance. Many interesting geometric properties of the parabolic trajectory were considered. Second, the computation of the path of a projectile by the method of differences (due to F. R. Moulton), which takes into account the most important types of resistance. Third, the computation of the path of a projectile by a new procedure called the functional method. This method requires no knowledge of the method of differences, and is comparatively simple both in theory and in practice.

2. *Applications of mathematics in aerodynamics*, by Professor R. Paul Harrington, Polytechnic Institute of Brooklyn, introduced by Professor R. M. Foster.

Two mathematical procedures which may be used in the solution of certain aerodynamic problems were discussed. The first dealt with the use of conjugate functions to represent certain physical characteristics of fluid flow. These flows were later transformed into curves (in another plane) representing airfoil cross-sections. The second dealt with the mathematical representation of a fuselage in an airstream, and the calculation of the effects upon the angle of attack of the propeller blades. The fuselage was represented by an ellipsoid of revolution, and the problem resolved iself into the determination of the coefficients in a series representing the velocity potential. The series was found to be composed of Legendre and associated Legendre functions so chosen that the surface of the fuselage became a stream line of the flow. The theoretical effects were certain velocity changes of such a nature as to alter the angle of attack of the blade by two or three degrees. This might result in torsional and bending vibrations of the blades.

3. *Combinatorial statistics*, by Dr. Jacob Wolfowitz, Columbia University, introduced by Professor Jewell Hughes Bushey.

The speaker remarked that the control of quality in articles produced by mass production offers an important field for the application of combinatorial statistics. He pointed out that sampling methods are in order whenever the cost of inspection makes complete inspection uneconomical or prohibitive. The Dodge-Romig sampling method was discussed from the viewpoint of modern statistical theory. The Shewhart method of runs for control of quality during manufacture was also described. A brief description of the theory of runs above and below the median, and runs up and down, was given. The asymptotic normality of the distribution of runs up and down will be proved by the speaker in a forthcoming article.

4. *A guiding philosophy for teaching demonstrative geometry*, by Mr. Morris Hertzig, Forest Hills High School, introduced by Mr. Max Peters.

Mr. Hertzig stated that the high school geometry course cannot be successfully taught as an abstract mathematical science. If taught as an empirical science of space, the following objectives can be realized: (1) the student has an opportunity to engage in experimental inquiries and to study the logic of sound experimental procedure; (2) he learns that an induction obtained as a generalization from several observations is insufficient to give scientific knowledge of the principle observed; (3) the student is given his only opportunity to study the complete scientific method (because high school courses are not organized with this objective in mind); (4) through a study of the deductive aspect of the scientific method the student develops understanding of an important device for extending the limits of knowledge.

5. *Mathematics and empirical science*, by Professor C. G. Hempel, Queens College, introduced by Professor T. F. Cope.

The speaker examined the significance of geometric theories as they pertain

to our knowledge of physical space. Two meanings of the term "geometry" were recognized, namely, pure geometry and physical geometry. A pure geometry, euclidean or non-euclidean, is an uninterpreted deductive system. It does not concern physical space, and the mathematical certainty of its theorems is due to the fact that each theorem simply re-asserts part of the content of the postulates. A system of physical geometry is obtained by assigning to each primitive of pure geometry its customary physical meaning. This transforms the postulates and theorems into physical hypotheses which may be said to concern the structure of physical space. When combined with the nongeometric part of physics, a physical geometry can be tested empirically. Recent physical findings appear to support the hypothesis, developed in connection with the general theory of relativity, that the geometric structure of the universe at large is of a certain non-euclidean type.

H. E. WAHLERT, *Secretary*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota section of the Mathematical Association of America was held at Macalester College in St. Paul, Minnesota, on Saturday, May 6, 1944. Sessions were held in the forenoon, at luncheon, and in the afternoon. Professor C. H. Gingrich, Chairman of the Section, presided.

Forty-five persons attended the meeting, including the following twenty-one members of the Association: R. W. Brink, W. E. Brooke, L. E. Bush, W. H. Bussey, E. J. Camp, C. S. Carlson, R. W. Erickson, I. C. Fischer, Gladys Gibbens, C. H. Gingrich, W. L. Hart, Dunham Jackson, C. M. Jensen, W. R. McEwen, Helen K. Milleson, J. M. H. Olmsted, Abraham Spitzbart, F. J. Taylor, H. L. Turrittin, A. L. Underhill, G. L. Winkelmann, and Sister Thomas à Kempis (institutional representative).

At the business meeting the following officers were elected for the coming year: Chairman, L. E. Bush, College of St. Thomas; Secretary, A. L. Underhill, University of Minnesota; Executive Committee, E. J. Camp, Macalester College, C. S. Carlson, St. Olaf College, H. L. Turrittin, University of Minnesota, K. W. Wegner, Carleton College.

The following seven papers were presented:

1. *Sophus Lie*, by Professor C. S. Carlson, St. Olaf College.

This paper was devoted to a biographical sketch of Sophus Lie. Particular attention was directed to his early childhood, his education, and the first few years following the completion of his education. An attempt was made to show how he came to enter upon his mathematical studies, and his mathematical career was summarized briefly.

2. *A note on the evaluation of* $\int_0^\infty \cos mx\,(1+x^2)^{-1}dx$, by Professor E. J. Camp, Macalester College.

The function $u(m, \beta)$ was defined by the equation

$$u(m, \beta) = \int_0^\infty \frac{e^{-\beta x} \cos mx}{1 + x^2} dx.$$

It was shown that this function satisfies the differential equation

$$\frac{d^2u}{dm^2} - u = -\frac{\beta}{\beta^2 + m^2}.$$

After solving this differential equation, it was found that

$$\lim_{\beta \to 0} u(m, \beta) = \int_0^\infty \frac{\cos mx}{1 + x^2} dx = \frac{\pi}{2} e^{-|m|}.$$

3. *Peacetime education in the light of the war training programs*, by Professor R. W. Brink, University of Minnesota.

4. *Remarks on activities of the committee on war programs of the American Mathematical Society and the Mathematical Association of America*, by Professor W. L. Hart, University of Minnesota.

5. *Elliptical and ellipsoidal boundary value problems*, by Professor Dunham Jackson, University of Minnesota.

6. *A study: Caroline Herschel*, by Sister M. Thomas à Kempis, College of St. Teresa.

This address was devoted to a brief study of Caroline Herschel, the first woman to discover a comet. The speaker described how Caroline Herschel shared the labors, hardships, and fatigues of her brother, William Herschel. Her perseverance and devotion made possible the distinctions enjoyed by her brother and by her nephew Sir John Herschel.

7. *A simple application of the density of the values of the complex exponential function* 1^a, by Professor J. M. H. Olmsted, University of Minnesota.

Professor Olmsted gave a proof that corresponding to any $\epsilon > 0$ and any irrational number a, there exist integers p and q such that

$$\left| a - \frac{p}{q} \right| < \frac{\epsilon}{q}.$$

The proof was developed from a special case of a simple consequence of the fact that, if a is an irrational number, then the values of 1^a are a set of points dense on the circumference of the unit circle.

A. L. UNDERHILL, *Secretary*

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Edited by OTTO DUNKEL, Washington University, ORRIN FRINK, JR., Pennsylvania State College, and H. S. M. COXETER, University of Toronto.

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